18.03SC Differential Equations, Fall 2011
Transcript – Damped Harmonic Oscillators

PROFESSOR: Welcome to this recitation on damped harmonic oscillators. So here you're asked to assume an enforced, overdamped spring-mass-dashpot that started at x dot of 0 close to 0, so rest, and to show that it never crosses the equilibrium position, x equal to 0, for t larger then 0.

The second part of the problem asks you to show that regardless of the initial condition, this overdamped oscillator can not cross the equilibrium position more than one time, or more than once.

OK. So why don't you pause the video, try to think about this problem, and I'll be right back.

Welcome back. So the system that we're looking at is a spring-mass-dashpot that would be written in this form-- second order differential equation. Let's assume that we have positive constant coefficients here.

And basically, to solve this, you would be considering the methods that we saw before. And the general solution would just be written in the form of c1 exponential lambda 1, the c2 exponential lambda 2, with c1, c2, just two constants that would be determined by the initial condition. Lambda 1, lambda 2 would be here the roots of the characteristic polynomial that you would have found there. Given that we're looking at an overdamped forced spring-mass-dashpot, you can actually show that lambda 1 and lambda 2 would be both real and negative. And here, we can just say that basically, lambda 1 and lambda 2 would be less than 1. And we'll keep that aside for now, and I'll use this later.

So this is just setting up the problem. So now, what was the question? The question was to show that if we start this system with initial condition x dot of 0 equal to 0, which corresponds to lambda 1 c1 plus lambda 2 c2 equals to 0, then the system cannot cross the equilibrium position x equal to 0 for t larger than 0.

So let's just start by assuming that the system crosses the equilibrium position. So for example, let's look for t star such that x of t star is equal to 0. So x of t star, we know its form already. We have the general form of x of t star. That's basically c1 exponential lambda 1 t star plus c2 exponential lambda 2 t star. And so we can massage this equation, and basically end up with minus c2 over c1 equal to exponential of lambda 1 minus lambda 2 t star.

So now let's just find our t star by applying the log of both sides of this equation. So we get t star equals to the log of minus c2 over c1, and we divide by the lambda 1 minus lambda 2.

So here this tells us that if t star exists, which means, if the log is defined, and this minus c2 over c1 basically is positive, then we only have one value of t star possible. So here, we actually are answering the second part of this question, number two, which was telling us that regardless of the initial condition-- so regardless of the coefficient c1 c2 that we would have-- if t star exists, there is only one. And so that means that the system would not cross this equilibrium position more than once. But I'll come back on that.

But now let's go back to what we were asked to do in the first part, where we basically now go back to our x dot of 0 equals to 0, which basically gave us that minus c_2 over c_1 is equal to lambda 1 over lambda 2, and the way we define lambda 1 over lambda 2 here gives us that minus c_2 over c_1 is less than 1, which means that the log is going to be negative.

What happens in the denominator? Lambda 1 minus lambda 2 would be positive. So with this initial condition, we would end up with a t star that would be negative. So basically, x is never equal to 0 again for t larger than 0, given these initial conditions.

So that finishes this first part of the problem. So I'll go back on the physics of it in a moment with a graph. So starting from this initial condition, x can never be equal to 0, because the only t star we can find would be negative. So for t larger than 0, it does not cross the equilibrium point.

The second part of the problem-- so this was 1-- just comes from the fact that, if I label this star, star tells us that if minus c_2 over c_1, strictly larger than 0, then only 1 t star exists.

So if we have a solution, there is only one. And this is regardless of the initial conditions that we would be given. So the system cannot cross the equilibrium position more than once.

So now let's look at what we're doing here, graphically. Let's assume that we're, for example, starting with initial condition here, where we're stretching our spring, but we start with 0 velocity. So x dot of 0 equal to 0. That was the system that we had. Then this is an overdamped case where both basically exponentials are decaying to 0, and so we would have a solution that would go to 0 quickly. It would be damped. And so this would be part 1.

Now let's look at what would happen if we started with other initial conditions. So for example, starting from the same point with a much bigger velocity. Then the system would go up, but eventually, it has to go down. It wouldn't have this shape, but basically, it would have to go down to 0's position.

And when it reaches, you can show that again, the derivative of x can reach 0 only once. And at that point, you're then back to the initial conditions that you had in the first part of the question, and so you can then, from here, argue again that you cannot cross the zero, the equilibrium point, after reaching a maximum.

Now what if we had a stretch that would be giving a negative velocity to the mass, and a very strong negative velocity? Then the system also wants to go back to 0, but could overshoot. And the overshoot would also generate a unique time at which the derivative would be equal to 0, and after that point, you would be back to the same argument we had before, where the solution would have to go toward 0, but never crosses it.

So we can have various configurations. And here I start with this point, but you could also start with other initial conditions, where you could have, as well, something that would be, for example, a very strong positive, where again, here you would have an overshoot, but then the solution would be attracted by the x equal to 0 solution.

And of course you could also start from the equilibrium. If you're not imposing any initial velocity, you just stay there, because this is not forced. But if you're imposing a velocity, then you would have other trajectories of the kind, for example, like this, where again, it would go up, but then be attracted back by the 0 solution.

So that's the typical behavior for a damped oscillator, where basically there's no oscillation, but the solution is attracted to rest. And you could have cases of overshoot when it can show that after the overshoot, velocity would reach 0 maximum, and then would be attracted back to the 0 solution with never crossing it. And that ends this recitation.

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