18.03SC Practice Problems 2

Direction fields, integral curves, isoclines, separatrices, funnels

Solution Suggestions

As an example, take the ODE

$$\frac{dy}{dx} = x - 2y.$$

1. *Draw a big axis system and plot some isoclines, especially the nullcline. Use them to illustrate the direction field. Using the direction field, plot a few solutions. Try to do this by hand first. Later you might want to refer to the* `Isoclines` *Mathlet.*

Do this on paper first. Then start the *Isoclines* applet in the *Mathlets Gallery*.

Set the equation to be $y' = x - 2y$. Move the m slider on the right to generate different isoclines (yellow lines), and click on the axis system to plot the solutions (blue curves). Compare the results with what you drew by hand.

2. *One of the integral curves seems to be a straight line. Is this true? What straight line is it? (i.e., for what m and b is $y = mx + b$ a solution?)*

Yes, one of the curves seems to be a straight line. You can guess visually from the applet that the linear solution seems to be the graph of the function $y = \frac{1}{2}x - \frac{1}{4}$. You can verify this guess is indeed a solution to the equation by checking that $\frac{dy}{dx} = \frac{1}{2} = x - 2(\frac{1}{2}x - \frac{1}{4}) = x - 2y$.

This answer can also be computed without guesswork. If $y = mx + b$ is a solution, then

$$m = \frac{dy}{dx} = x - 2y = x - 2(mx + b),$$

or, rearranging terms,

$$m = (1 - 2m)x - 2b.$$

Two polynomials in x can only the same for all x (over the reals) if they have the same coefficients. Equate coefficients of the powers of x to get two equations in two unknowns that must be satisfied simultaneously.

$$\begin{cases} m = -2b \\ 0 = 1 - 2m \end{cases}$$

From the second equation, $m = \frac{1}{2}$. Plugging this into the first equation gives $b = -\frac{1}{4}$. This matches the answer guessed from the applet.

3. *In general – for the general differential equation $\frac{dy}{dx} = F(x, y)$ – if a straight line is an integral curve, how is it related to the isoclines of the equation? What happens in our example?*

If a straight line is an integral curve, then it is part of the isocline for its slope. In our example, the line from Question 2 forms the entire $m = \frac{1}{2}$-isocline.

4. *It seems that all the solutions become asymptotic as $x \to \infty$. We will see later that this is true, but for now, explain why solutions get trapped between parallel lines of fixed slope.*

Here all the isoclines have the form $m = x - 2y$ for some constant m. That is, the isoclines are lines of slope $\frac{1}{2}$. When $m > \frac{1}{2}$, any solution through a point on an m-isocline will have slope of tangent line ($\frac{dy}{dx} = F(x,y) = m$) greater than the slope of the linear isocline ($\frac{1}{2}$) at the point of intersection. Thus, any solution that intersects such an isocline must pass from the region below the isocline to the region above it. In particular, no solution can cross in the other direction. Similarly, any solution through a point on an m-isocline for $m < \frac{1}{2}$ must pass from the region above the line to the region below it.

From the above, the m-isocline is given by the equation $y = \frac{1}{2}x - \frac{m}{2}$, so all the isoclines for $m > \frac{1}{2}$ lie below the line $y = \frac{1}{2}x - \frac{1}{4}$, while all the isoclines for $m < \frac{1}{2}$ lie above it. Thus, any solution will eventually get trapped between any two isoclines that surround the line $y = \frac{1}{2}x - \frac{1}{4}$, i.e. between parallel lines of fixed slope $\frac{1}{2}$.

5. *Where are the critical points of the solutions of $y' = x - 2y$? How many critical points can a single solution have? For what values of y_0 does the solution y with $y(0) = y_0$ have a critical point? When there is one, is it a minimum or a maximum? You can see an answer to this from your picture. Can you also use the second derivative test to be sure?*

All critical points lie on the $y' = 0$-isocline, which is the line $y = \frac{1}{2}x$. We will argue that solutions above the linear integral curve $y = \frac{1}{2}x - \frac{1}{4}$ found in Question 2 have exactly one critical point and solutions below it have no critical points. For graphical intuition, refer to the following screenshot of the Isoclines applet with a picture of the direction field for this equation, the nullcline, the linear integral curve, and several solutions.
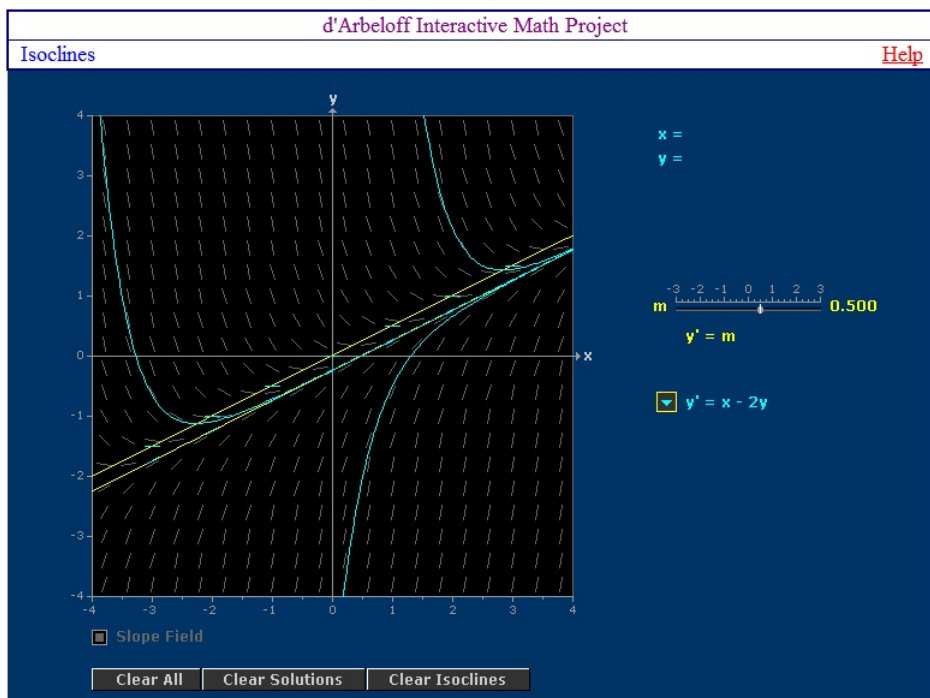

Figure 1: Several isoclines and solutions for the equation $y' = x - 2y$

The question can be rephrased as how do solutions touch or intersect the nullcline.

From the graph it seems that the solution space is divided in two, and the only solutions to intersect the nullcline are those that lie above the $\frac{1}{2}$-isocline $y = \frac{1}{2}x - \frac{1}{4}$. From Question 4, the $\frac{1}{2}$-isocline is an asymptote, and solutions stay above this line for all x if they are above it at any point – for example, if they have a larger y-intercept ($y_0 > -\frac{1}{4}$). We will show that solutions have a critical point exactly when they are contained in the upper solution space.

One direction of this is easy. All the critical points lie on the nullcline, which is in the upper solution space, so a solution has a critical point only if it also lies in the upper solution space. To show the other direction, use the geometry of the isoclines that we found - that these are parallel lines of slope $\frac{1}{2}$, of steadily increasing value equal to negative double their y-intercept. So solutions in the upper solution space have translational symmetry and all look like the solution through $(0,0)$, shifted along the nullcline. In particular they all intersect the nullcline eventually.

Finally, since the nullcline is also a line of slope $\frac{1}{2}$, any solution going through a point on it must pass from the region to the left of it to the region to the right of it, by the same reasoning as in Question 4, since at the point of intersection $\frac{dy}{dx} = 0 < \frac{1}{2}$. Therefore, all solutions which intersect the nullcline at all will pass through it exactly once, so they have exactly one critical point.

To summarize, solutions can have either zero or one critical points, and a solution has a critical point if and only if it lies above the linear integral curve $y = \frac{1}{2}x - \frac{1}{4}$ (has y-intercept $y_0 > -\frac{1}{4}$).

The picture also makes it appear that all of the critical points are minima.

We can verify this by using the second derivative test. Find an expression for the second derivative by taking the derivative of both sides of the differential equation:

$$\frac{d^2y}{dx^2} = 1 - 2\frac{dy}{dx}.$$

At a critical point, $\frac{dy}{dx} = 0$ by definition. Plug this into the above equation to get that $\frac{d^2y}{dx^2} = 1$ at any critical point. The second derivative is positive, so any critical point must be at least a local minimum. We just showed that a solution has at most one critical point, so it must be a global minimum.

6. *For another example, take $\frac{dy}{dx} = y^2 - x^2$. (This is also on the* `Isoclines` *Mathlet.) Again, make a big picture of some isoclines and use them to sketch the direction field, and then sketch a few solutions.*

Set the equation to be $y' = y^2 - x^2$ in the applet, and follow the same procedure as in question 1.

7. *A "separatrix" is a solution such that solutions above it have a fate (as x increases) entirely different from solutions below it. The equation $\frac{dy}{dx} = y^2 - x^2$ exhibits a separatrix. Sketch it and describe the differing behaviors of solutions above it and below it.*

Play with the applet to get a feeling for the solution space. You should find that here the solutions either increase without bound or decrease without bound. You can use the applet to visually approximate the separatrix between these two fates by

drawing better and better distinguishing solutions. You can do this, for example, by choosing points near the top right of the picture (e.g., click near (3.70, 3.84)).

After sketching the separatrix, describe the behaviors of solutions above and below it. Solutions above the separatrix rapidly increase without bound. Solutions below it approach the lower right asymptote of the -1-isocline, which forms the hyperbola $y^2 - x^2 = -1$.

Some justification for this behavior is that the line $y = x$ is part of the nullcline, and the direction field above it increases rapidly (is quadratic), while the direction field below it decreases rapidly.

8. *The equation $\frac{dy}{dx} = y^2 - x^2$ also exhibits a "funnel," where solutions get trapped as x increases, and many solutions are asymptotic to each other. Explain this using a couple of isoclines. There is a function with a simple formula (not a solution to the equation, though) which all these trapped solutions get near to as x gets large. What is it?*

We saw in the previous question that solutions below the separatrix approach the lower right asymptote of the -1-isocline - the hyperbola $y^2 - x^2 = -1$, whose lower right asymptote is the line $y = -x - 1$. You can use the applet to observe further that solutions approach the asymptote quickly, becoming trapped between isocline fences, which themselves quickly approach this asymptote. Move in some isoclines around the asymptote and see how they force solutions to behave.

For example, the lower part of the -2-isocline (the negative part of the hyperbola $y^2 - x^2 = -2$, which lies above and to the right of $y^2 - x^2 = -1$) seems to act as an upper fence for large enough values of x, forming the upper part of a funnel. Verify this by checking for which values of x the derivative of this isocline becomes greater than -2. Observe first that the m-isoclines in this this example are hyperbolas $y^2 - x^2 = m$ with slope x/y at each point (x, y). Then solve the inequality $x/y > -2$ subject to the restriction $y^2 = x^2 - 2$ to see that these conditions are satisfied simultaneously in the fourth quadrant exactly when $x > \sqrt{\frac{8}{3}} \approx 1.64$.

In turn, for example, the lower half of the $\frac{1}{2}$-isocline (the lower half of the hyperbola $y^2 - x^2 = \frac{1}{2}$, which lies below and to the left of $y^2 - x^2 = -1$) seems to act as a lower fence for large enough values of x, forming the lower part of a funnel. To verify this, check when the derivative of this isocline becomes less than 1/2. Solving the inequality $x/y < \frac{1}{2}$, subject to $y^2 = x^2 + \frac{1}{2}$, see that these conditions are satisfied simultaneously for all (x, y) in the fourth quadrant (in fact, for all $x > -\sqrt{\frac{1}{6}} \approx -0.41$).

You can use the applet to visually check the values we found for the points at which each isocline becomes a fence. Below is a screenshot of the applet displaying our direction field, together with these two isoclines and some solutions that intersect them.
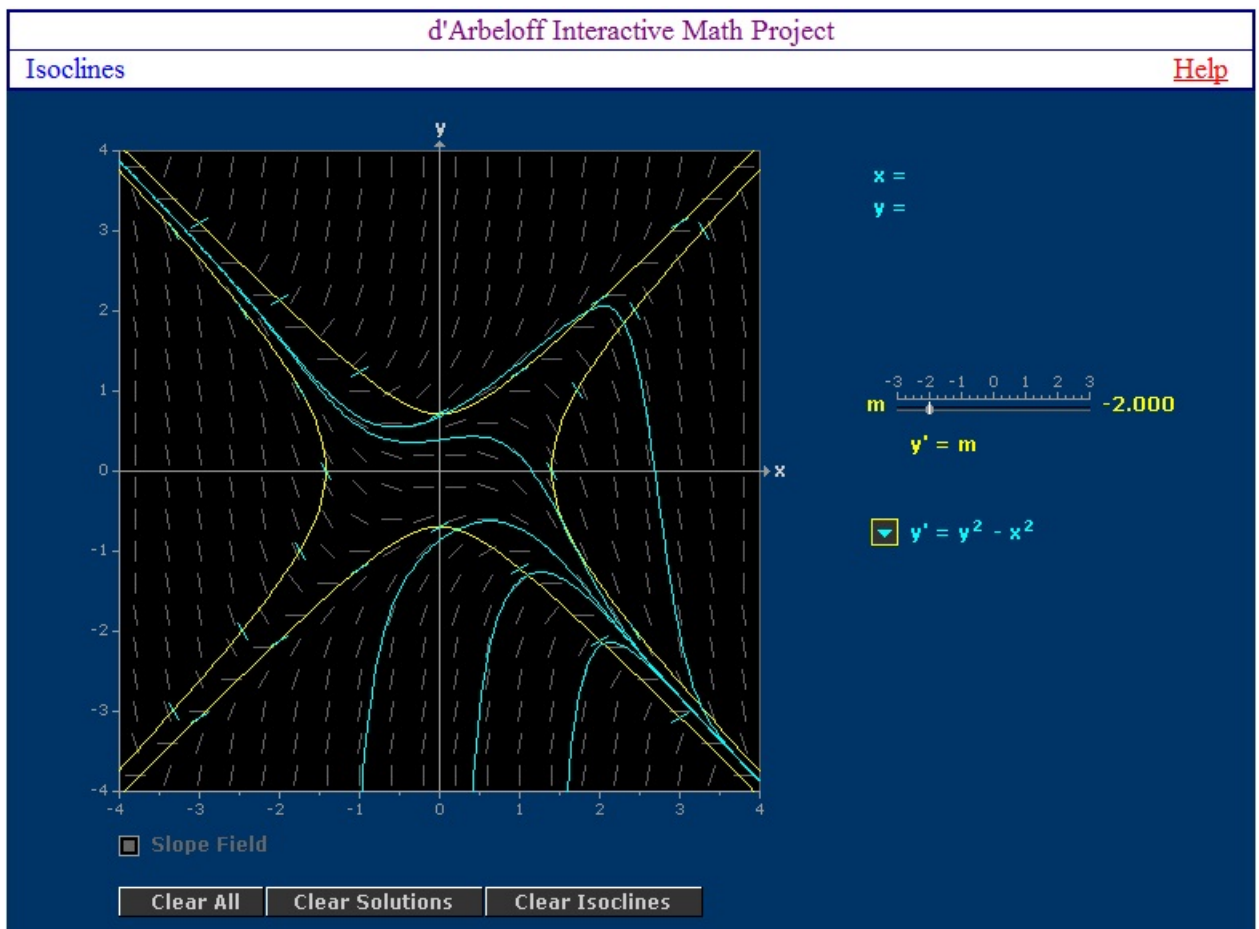
Figure 2: The direction field for $\frac{dy}{dx} = y^2 - x^2$, some solutions and a funnel.

MIT OpenCourseWare
http://ocw.mit.edu

18.03SC Differential Equations
Fall 2011

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.