18.034 Honors Differential Equations
Spring 2009

LECTURE 28. REPEATED EIGENVALUES AND THE MATRIX EXPONENTIAL

Repeated eigenvalues. Again, we start with the real $n \times n$ system

$$\vec{y}\,' = A\vec{y}. \tag{28.1}$$

We say an eigenvalue λ_* or A is *repeated* if it is a multiple root of $p_A(\lambda)$. That is, $p_A(\lambda)$ has $(\lambda - \lambda_*)^2$ as a factor. We suppose that λ_* is a double root of $p_A(\lambda)$, so that in principle two linearly independent solutions of (28.1) correspond to λ_*. If $\vec{v}_*$ is the corresponding eigenvector, then $\vec{y} = \vec{v}_* e^{\lambda_* t}$ is a solution. The problem is to find the second solution of (28.1), linearly independent of $\vec{y}_*$. We first discuss an easy case.

Example 28.1 (The complete case)**.** Still supposing that λ_* is a double root of $p_A(\lambda)$, we say λ_* is a *complete* eigenvalue if there are two linearly independent eigenvectors corresponding to λ_*, say $\vec{v}_1$ and $\vec{v}_2$. Using them, we obtain two linearly independent solutions of (28.1), namely

$$\vec{y}_1(t) = \vec{v}_1 e^{\lambda_1 t} \quad \text{and} \quad \vec{y}_2(t) = \vec{v}_2 e^{\lambda_1 t}$$

Exercise. Let A be a 2×2 matrix. If λ is a repeated and complete eigenvalue of A, show that

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

The converse holds true.

In general, if an eigenvalue λ_* of A is k-tuply repeated, that is, $p_A(\lambda) = |A - \lambda I|$ has the power $(\lambda - \lambda_*)^k$ as a factor, but no higher, the eigenvalue is called *complete* if k linearly independent eigenvectors correspond to λ_*. In this case, these eigenvectors produce k linearly independent solutions of (28.1). An important result in linear algebra is that if a real square matrix A is symmetric, that is, $A = A^T$, then all its eigenvalues are real and complete.

However, in general, an eigenvalue of multiplicity k (> 1) has less than k eigenvectors, and we cannot construct the general solution from the techniques of eigenvalues.

The matrix exponential. To motivate, the initial value problem

$$y' = ay, \qquad y(0) = 1$$

has the solution $y(t) = e^{at}$ in the form of exponential. We want to define the expression e^{At} for a general $n \times n$ matrix A, $n \geqslant 1$, so that $Y(t) = e^{At}$ is a solution of

$$Y' = AY, \qquad Y(0) = I$$

and moreover e^{At} is easy to compute.

Recall that if a is a real number then

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots + \frac{(at)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(at)^n}{n!},$$

where $t \in \mathbb{R}$. A natural way to define the matrix exponential e^{At}, for an $n \times n$ matrix A, seems to use the series expression

$$e^{At} = I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^nA^n}{n!}. \tag{28.2}$$

In order to make sense of the above expression, we first introduce the matrix norm.

Definition 28.2. For an $n \times n$ matrix A, define the *matrix norm* as

$$\|A\| = \sup_{\vec{y} \neq 0} \frac{|A\vec{y}|}{|\vec{y}|},$$

where $|\vec{y}| = |\vec{y}^T\vec{y}|^{1/2} = (y_1^2 + \cdots + y_n^2)^{1/2}$ and $|A\vec{y}| = |(A\vec{y})^T(A\vec{y})|^{1/2}$.

Exercise. Show that

$$\|A + B\| \leqslant \|A\| + \|B\|, \qquad \|AB\| = \|A\|\,\|B\|, \qquad \|tA\| = |t|\,\|A\|$$

for any matrices A, B and for any $t \in \mathbb{R}$.

For a matrix-valued function $A(t) = (a_{ij}(t))_{i,j=1}^n$, then $A(t) \to A(t_0)$ means equivalently:

(i) $a_{ij}(t) \to a_{ij}(t_0)$ as $t \to t_0$ for all $1 \leqslant i, j \leqslant n$.
(ii) $||A(t) - A(t_0)|| \to 0$ as $t \to t_0$.

Under the matrix norm, the infinite series (28.2) converges for all A and for all t, and it defines the *matrix exponential*.

We now compute the derivative of e^{At} by differentiating the right side of (28.2) term by term

$$\begin{aligned}\frac{d}{dt}e^{At} =& \frac{d}{dt}\left(I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots\right)\\ =& A + tA + \cdots + \frac{t^{n-1}A^n}{(n-1)!} + \cdots\\ =& Ae^{At}.\end{aligned}$$

Moreover, since $e^{A0} = I$, by definition, the matrix exponential e^{At} is a solution of

$$Y' = AY, \qquad Y(0) = I.$$

Theorem 28.3. *Let $Y(t)$ be a fundamental matrix of A. Then $Y(t) = Y(0)e^{At}$.*

For several classes of A, the infinite series in (28.2) can be summed up exactly.

Exercise. 1. Show that $\exp(\text{diag}\,(\lambda_1 \ \ldots \ \lambda_n)) = \text{diag}\,(e^{\lambda_1} \ \ldots \ e^{\lambda_n})$.

2. Show that if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $e^{At} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$.

Exercise. Prove the exponential law

$$e^{(A+B)t} = e^{At}e^{Bt} \qquad \text{if} \quad AB = BA. \tag{28.3}$$

We use (28.3) to compute the matrix exponentials of more matrices.

Example 28.4. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. We write $A = B + C$, where

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $BC = CB$, by the results from the previous exercise and by (28.3)

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Finding the fundamental matrix. In general, it doesn't seem possible to express e^{At} in closed form. Nevertheless, we can find n independent vectors for which e^{At} can be computed exactly.

The idea is to write

$$e^{At}\vec{v} = e^{(A-\lambda I)t}e^{\lambda It}\vec{v} = e^{(A-\lambda I)t}e^{\lambda t}\vec{v}.$$

If $(A-\lambda I)^m\vec{v} = 0$ for some integer $m > 0$, then $(A-\lambda I)^{m+l}\vec{v} = 0$ for all integers $l \geqslant 0$. Hence,

$$e^{(A-\lambda I)t}\vec{v} = \vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}$$

is a finite sum, and

$$e^{At}\vec{v} = e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\vec{v}\right)$$

can be computed exactly, although e^{At} itself cannot be computed.

Example 28.5. Solve the system $Y' = AY$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

SOLUTION. Its characteristic polynomial is $p_A(\lambda) = (1-\lambda)^2(2-\lambda)$, and it has a double root $\lambda = 1$ and a simple root $\lambda = 2$.

If $\lambda = 1$, then

$$(A-\lambda I)\vec{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{v} = 0$$

has a solution $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Furthermore, it is the only eigenvector for $\lambda = 1$ up to a constant multiple. (In fact, the well-known result from linear algebra tells us that the solution space of the above linear system of equations has dimension 1.) One solution of the system $Y' = AY$ is obtained and $\vec{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

To find the second solution associated to $\lambda = 1$, we compute

$$(A-\lambda I)^2\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{v} = 0$$

has a nontrivial solution $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, linearly independent to v_1. On the other hand, $(A-I)\vec{v}_2 \neq 0$.

Thus,

$$e^{At}\vec{v}_2 = e^t(I + t(A-I)\vec{v}_2) = e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix},$$

and it gives an additional solution of the system $Y' = AY$ associated to $\lambda = 1$.

Finally, if $\lambda = 2$ then

$$(A - 2I)\vec{v} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = 0$$

has a solution $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Hence, $\vec{y}_3(t) = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The general solution of the system $Y' = AY$ can be written as

$$Y(t) = (c_1 + tc_2)e^t \vec{v_1} + c_2 e^t \vec{v_2} + c_3 e^{2t} \vec{v_3}.$$

It is analogous to that for the scalar differential equations with multiple roots.

We conclude the lecture by the following important result in linear algebra.

Theorem 28.6 (Cayley-Hamilton Theorem)**.** *Any square matrix A satisfies $p(A) = 0$, where p is the characteristic polynomial of A.*

Proof. Recall the formula

$$\text{adj}\,(A - \lambda I) \cdot (A - \lambda I) = p(\lambda)I.$$

Both sides are polynomial expressions for $\lambda \in \mathbb{R}$. We view them as matrix polynomials, that is to say, we replace λ by a matrix. By setting $\lambda = A$ then proves the assertion. $\square$