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18.034 Honors Differential Equations
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LECTURE 24. THE TRANSFER FUNCTION AND THE POLE DIAGRAM

The transfer Function. Throughout this section

$$P(D) := D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,$$

where a_j are real constant. Differential equations of the form $P(D)y = f$ are widely used to representing electric alternating current networks or filters. Such a filter may be thought of as a “black box” into which an electric current or a voltage is fed as an input $f(t)$ and out of which comes a resulting output $y(t)$.

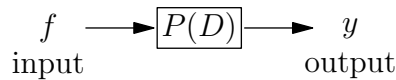


Figure 24.1. The input-output system.

Mathematically, it amounts to studying an operator transforming the input function f into a function y , the solution of $p(D)y = f$. Let us write formally $y = W[f]$. Then, $P(D)W[f] = f$. That is to say, such an input-output operator is a right inverse of $P(D)$.

In the simplest case, let $f(t) = e^{at}$. Under the assumption $P(a) \neq 0$,

$$(24.1) \quad y(t) = \frac{1}{P(a)} f(t) \quad \text{or} \quad W(a) = \frac{1}{P(a)}.$$

Such a function transforming the input signal to the output is called the *transfer function*. The main property of the transfer function is

$$(\text{output}) = (\text{transfer function}) \cdot (\text{input}).$$

Exercise. For a sinusoidal input $f(t) = e^{i\omega t}$, where $\omega \in \mathbb{R}$, show that $W(\omega) = 1/P(i\omega)$ unless $P(i\omega) = 0$. (The case $P(i\omega) = 0$ corresponds to resonance.)

A right inverse W of $P(D)$ is not defined unambiguously. For, $P(D)y = f$ does not have uniqueness. For initial value problems, one often defines W by imposing the null initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$.

For a broad class of inputs and outputs, we can characterize them by their Laplace transforms, and it naturally gives (Laplace transforms of) transfer functions. In fact, if $f \in E$, the rest solution of $p(D)y = f$ satisfies

$$(24.2) \quad P(s)Y(s) = F(s) \quad \text{or} \quad Y(s) = W(s)F(s),$$

where $W(s) = 1/P(s)$ is the transfer function.

Note that the second equation in the above says that

$$(\text{output}) = (\text{transfer function}) \cdot (\text{input}),$$

but the multiplication is in the s -domain.

A major advantage of (24.2) is that it applies to an arbitrary $f \in E$. This increased scope of the transfer function is one of the advantages of working in the frequency domain.

Application: feedback. The transfer function is useful when two or more systems are connected together so that the output of one system becomes the input of another system.

In the simplest case consider the upper system in Figure 24.2. The coupled system is described by

$$Y_1 = W_1(s)Y_0, \quad Y_2 = W_2(s)Y_1.$$

Hence,

$$Y_2 = W_2(s)W_1(s)Y_0.$$

That means, the upper system in Figure 24.2 is equivalent to the lower system. The transfer function for the whole system is the product of the individual transfer functions.

A more complicated system is shown in Figure 24.3. Here, the basic system is the one with W_1 and the system corresponding to W_2 provides a feedback to the system W_1 . Hence, the total input to W_1 is

$$Y_1 = Y_0 + W_2(s)Y_2.$$

On the other hand, the feedback system is described as

$$Y_2 = W_1(s)Y_1.$$

Eliminating Y_1 , we find a description of the total system

$$Y_2 = W_2(s)(Y_0 + W_2(s)Y_2), \quad Y_2 = (1 - W_1(s)W_2(s))^{-1}W_1(s)Y_0.$$

Therefore, the upper system is equivalent to the lower system associated with the transfer function $(1 - W_1(s)W_2(s))^{-1}W_1(s)$.

Application: identification. In the problem of identification, you put in a known input, measure the output, and determine the parameters in the differential equation. It is a common point of view in experimental sciences; after measuring the response to a known input, one can try to discover the mechanism by which the response was produced.

When the system in study takes the form $P(D)y = f$, where the differential operator is determined by its characteristic polynomial, this characteristic polynomial is determined by its transfer function $W(s) = 1/P(s)$. Thus, the problem of identification reduces to one of finding the transfer function.

In the very simplest case when the output is the unit impulse $f(t) = \delta(t)$, then $F(s) = 1$ and $Y(s) = W(s)$. The corresponding $y(t)$ is called the *unit impulse response*. In other words, the unit impulse response is the transfer function.

In the next simplest case when $f(t) = h(t)$ is the unit step function, then its Laplace transform is $F(s) = \frac{1}{s}$ and

$$Y(s) = \frac{W(s)}{s}.$$

The corresponding $y(t)$ is called the *unit step response*.

Example 24.1. Let

$$P(D) = D^2 + aD + bI,$$

and the unit impulse response is $y(t) = e^{2t} \sin t$, $t > 0$. Determine the constants a, b .

SOLUTION. In this case $\mathcal{L}[e^{2t} \sin t] = W(s) = \frac{1}{s^2 + as + b}$. On the other hand, $\mathcal{L}[e^{2t} \sin t] = \frac{1}{(s - 2)^2 + 1}$. Therefore, $a = 4$ and $b = 5$.

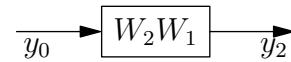
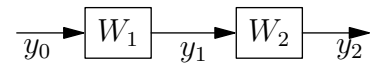


Figure 24.2.

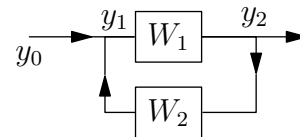


Figure 24.3

The pole diagram. The real power of the Laplace transform is not so much in algorithms for explicitly computing the responses to the systems as in gaining insight into the systems *without* explicitly computing the responses. For this purpose, it is better to think of the s -variable in $F(s)$ as ranging over the *complex* numbers.

In complex analysis, a function $F(s)$, where $s \in \mathbb{C}$, is said to have a *pole* at z if $(s - z)^n F(s)$ is holomorphic at z . The smallest positive integer n is called the order of the pole. If $n = 1$, then it is called a simple pole.

A basic example is $F(s) = 1/(s - z)$, for a fixed complex number z . Its magnitude $1/|s - z|$ is a real-valued function on the complex plane, and its graph is a surface lying over the complex plane, whose height over a point s is given by $1/|s - z|$. This is a tent-like surface lying over the complex plane, with elevation given by the reciprocal of the distance to z . It sweeps up to infinity like a hyperbola as $s \rightarrow z$, as if the graph is held up at $s = z$ by tent-pole.

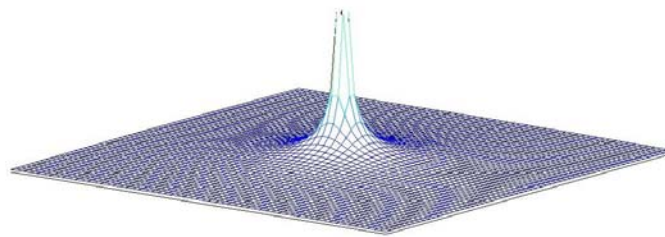


Figure 24.4. The graph of $1/(s - z)$.

A rational function $F(s)$ which has only simple poles can be decomposed into its partial fractions as

$$F(s) = p(s) + \frac{w_1}{s - z_1} + \cdots + \frac{w_k}{s - z_k},$$

where $p(s)$ is a polynomial, $z_j, w_j \in \mathbb{C}$; z_j are (simple) poles of $F(s)$ and $w_j = \lim_{s \rightarrow z_j} (s - z_j)F(s)$ is called the *residue* of the pole z_j . For example,

$$\frac{1}{s^3 + s^2 - 2} = \frac{1/5}{s - 1} + \frac{(-1 + 2i)/10}{s - (-1 + i)} + \frac{(-1 - 2i)/10}{s - (-1 - i)}.$$

The *pole diagram* of a complex function $F(s)$ is the complex plane with the poles of $F(s)$ marked on it. For example, the pole diagram of $\frac{1}{s^3 + s^2 - 2}$ is below.

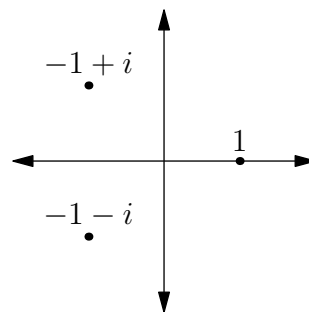


Figure 24.5. The pole diagram of $\frac{1}{s^3 + s^2 - 2}$.

The pole diagram of $F(s)$ tells us about long-term behavior of $f(t)$. (Laplace transforms we typically treat are rational functions with simple poles only. But, they are not always rational functions, e. g. $\mathcal{L}[\delta(t - c)] = e^{-sc}$ is an entire function.)

To illustrate it, we consider the functions

$$F_1(s) = \frac{c}{s-1}, \quad F_2(s) = \frac{ce^{-as}}{s-1}, \quad F_3(s) = \frac{c}{s-1} + b\frac{1-e^{-as}}{s},$$

where $a, b \in \mathbb{R}$ and $c \neq 0$. All three functions have two poles at $s = 1$. (Note that $1 - e^{-as} = 0$ at $s = 0$, canceling the zero in the denominator of the second term in F_3 .)

We assume that $a \geq 0$. They are the Laplace transforms of

$$f_1(t) = ce^t, \quad f_2(t) = ch(t-a)e^{t-a}, \quad f_3(t) = ch(t-a)e^t + bh(t-a).$$

All these functions grow like a multiple of e^t when t is large. You can even say which multiple; it is given by the residue at $s = 1$. But, their behavior when $t < a$ is all over the map.

Next, we consider the functions

$$F_1(s) = \frac{c(s-a)}{(s-a)^2 + b^2}, \quad F_2(s) = \frac{cb}{(s-a)^2 + b^2}.$$

They have two poles at $s = a \pm ib$, and they are the Laplace transforms of

$$f_1(t) = ce^{at} \cos(bt), \quad f_2(t) = ce^{at} \sin(bt).$$

This says that the *real part* of the pole determines the long term growth in maximum magnitude, and the *imaginary part* of the pole determines the frequency of oscillation for large t . Again, the pole diagram says nothing about the function for small t .

We combine these observations and study

$$F(s) = \frac{1}{s^3 + s^2 - 2} = \frac{1/5}{s-1} + \frac{(-1+2i)/10}{s-(-1+i)} + \frac{(-1-2i)/10}{s-(-1-i)}.$$

The function has poles at $s = 1$, $s = -1 \pm i$, and we expect that the anti-transform $f(t)$ has a term which grows like e^t (corresponding to the pole at $s = 1$) and another term which behaves like $e^{-t} \cos t$ (up to phase shifts). When t is large, the term from the *rightmost pole*, the one with the largest real part, dominates the behavior of $f(t)$. In fact,

$$f(t) = \frac{1}{5}e^t + \frac{-1+2i}{10}e^{(-1+i)t} + \frac{-1-2i}{10}e^{(-1-i)t}$$

grows like $\frac{1}{5}e^t$ for large t .

In summary, the position of the rightmost poles of $F(s)$ determines the general behavior of $f(t)$ for $t \gg 1$. If the rightmost pole is at $a + bi$, then $f(t)$ will behave roughly like a multiple of $e^{(a+ib)t}$. That is, it grows like e^{at} and oscillates like $\cos bt$.