18.034, Honors Differential Equations
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Lecture 3: Existence + Uniqueness, Part I

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1. Terminology: Let S be a subset of the plane $\mathbb{R}^2$. An interior point of S is a point (t_0,y_0) for which there exits a, b>0 (depending on S and (t_0,y_0)) such that $(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b)$ is contained in S. The notation $(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b)$ is shorthand for the set of all pairs (t,y) such that $t_0 - a < t < t_0 + a$ and $y_0 - b < y < y_0 + b$.

   A limit point of S is a point (t,y) that is the limit of a convergent sequence $(ti, yi)_{i=0,1,2,\ldots}$ every term of which is in S. Each point (t,y) of S is a point of S since it is the limit of the constant sequence : $(t_i, y_i) = (t, y)$ for all $i = 0,1,2,\ldots.$.

   The interior of S is the collection of all interior points of S. The closure of S is the set of all limit points of S. According to your textbook, a region is a set that is equal to its interior. This is different than the def'n I gave in lecture (and which many mathematicians use), that a region is a set R that is contained in the closure of its interior. According to this terminology, an "open region" is a set that is equal to its interior, and a "closed region" is a set that is equal to the closure of its interior. However, I will follow the notation in the book from here on.

   Let R be a region (= "open region") in $\mathbb{R}^2$ and let $D \subset R$ be a bounded, closed region. In other words, (1) D is a subset of R;
   (2) there exists N>0 such that D$D \subset [-N, N] \times [-N, N]$,
   (3) D is the closure of its interior.

   Also, since it doesn't follow automatically from the above, we usually also demand that D is not empty.

   Let f(t,y) be a real-valued function defined on R. Let (t_0,y_0) be an interior point of D.

   Def'n: A solution of the IVP
   $$\begin{cases} y(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

   defined in D is a pair $(I, y(t))$ consisting of an interval $I = (a, b)$ and a real-valued function y(t) such that
   (0) t_0 is in (a, b)
   (1) the graph of $y(t)$ is contained in the interior of I
   (2) $y(t)$ is differentiable
   (3) for all t in (a, b), $y'(t) = f(t, y(t))$
   (4) $y(t_0) = y_0$

   A solution (I, y) is called a maximal solution (or a maximally-extended solution) if for every solution (I_1, y_1), the interval I_1 is contained in I_1 and y_1 is the restriction of y to I_1.

2. The main theorem is this.f(t,y)

   Thm [=Thms A.1.1, A.2.3, A.3.1] Let R be a region in the plane, let D$D \subset R$R be a bounded, closed region, let $f(t, y)$ be defined on R, and let (t_0,y_0) be an interior point of

D. If $f(t,y)$ is continuous on R, and if the partial derivative $\frac{\partial f}{\partial y}$ is everywhere defined and continuous on R, then there exists a maximal solution (I,y) of the IVP:

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

Moreover, denoting $I = (a,b)$, $\lim_{t\to a} y(t)$ and $\lim_{t\to b} y(t)$ exists--- call them $y(a)$ & $y(b)$--- and $(a, y(a))$, $(b, y(b))$ are <u>boundary points of D</u> i.e points of D that are not interior points.

This statement is a bit complicated. What it really says is this:

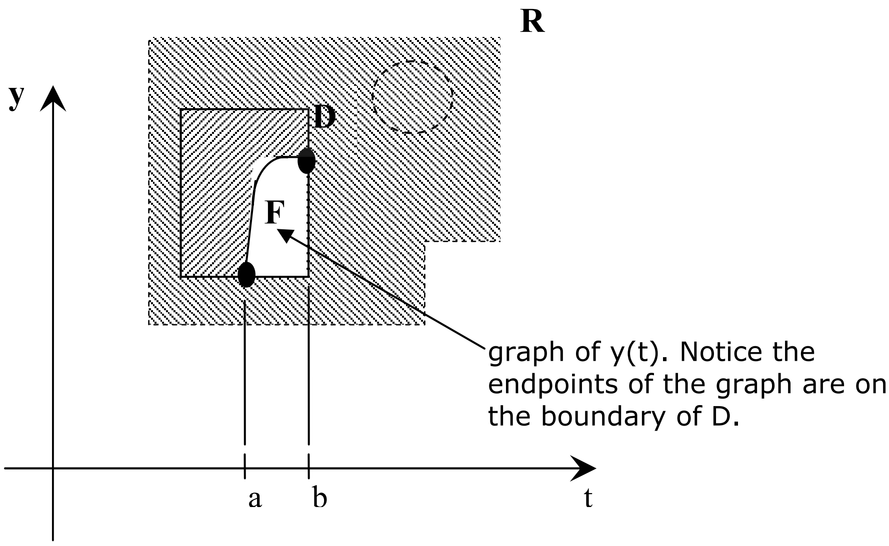
(1) <u>Existence</u>: There exists an interval (a,b) contains t_0 and a solution y(t) of the IVP defined on (a,b).

(2) <u>Uniqueness</u>: If $y_1(t)$ is any solution of y(t) defined on (a,b), then $y_1(t)=y(t)$ for all t in (a,b).

(3) <u>Maximal extendability</u>: A formal consequence of (1) & (2) is that there exists a solution where (a,b) is the largest possible interval on which any solution is defined.

(4) <u>Endpoints of y</u>: The endpoints of the graph of y(t) lie on the boundary of D.

<u>Picture</u>:



The proof of the theorem proceeds in several stages. Most of the proof is carried out after replacing D by a small rectangle containing (t_0,y_0) (at the end, it will be shown how to deduce the theorem from this case).

3. Shrinking D_i Lipschitz condition.

Because (t_0,y_0) is an interior point of D, there exists $a, b > 0$ such that $(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b) \subset D$. Because D is closed, also the <u>closed rectangle</u> $(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b)$ is contained in D.

The <u>maximum principle</u> states that any continuous function on a bounded, closed region attains a (finite) maximum at some point in the closed region. It also attains its minimum at some point. As a consequence, there exists M>0 such that for all (t,y) in $(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b)$, $|f(t,y)| < M$.

Similarly, because $\frac{\partial f}{\partial y}$ is defined and continuous, there exists L>0, such that for all (t,y) in

$(t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b)$, $\left|\frac{\partial f}{\partial y}(t,y)\right| < L$.

This last inequality has a useful consequence.

<u>Def'n</u>: A function f defined on a rectangle $[t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ is a <u>Lipschitz with respect to y</u>, <u>with Lipschitz constant L</u>, if for every (t,y_1) and (t,y_2) in the rectangle, $|f(t,y_2) - f(t,y_1)| \le L|y_2 - y_1|$.

The function f(t,y) is Lipschitz with respect to y, with Lipschitz constant L. To see this, note that by the mean value theorem, $f(t,y_2) - f(t,y_1) = \frac{\partial f}{\partial y}(t,y_3).(y_2 - y_1)$ for some y_3 in (y_1, y_2).

Therefore, since $\frac{\partial f}{\partial y}(t,y_3) \le L$, we have $|f(t,y_2) - f(t,y_1)| \le L|y_2 - y_1|$ for all (t,y_1) and (t,y_2) in the rectangle.

Let c>0 be any positive number less than $\min\left\{a, \frac{b}{M}, \frac{1}{2L}\right\}$. Observe that each of $a, \frac{b}{M}, \frac{1}{2L}$ is positive, so also c is positive. The rectangle we work with is $[t_0, t_0 + C]\,X\,[y_0 - b, y_0 + b]$.

4. Metric spaces; Contraction mappings.

There is a useful language, sometimes not introduced until 18.100, of <u>metric spaces</u>. The idea is to identify and make abstract all properties of Euclidean space that can be stated and proved using only the distance function.

<u>Def'n</u>: A <u>metric space</u> is a pair (X, ∂) consisting of a set X and a real-valued function $\partial(p,q)$, defined for all pairs p,q of elements of X, and that satisfies:

(0) for all p,q , $\partial(p,q) \ge 0$
(1) $\partial(p,q) = 0$ iff (= "if and only if") p=q
(2) for all p,q, $\partial(p,q) = \partial(q,p)$ (i.e. ∂ is "symmetric")
(3) for all p,q and r, $\partial(p,r) \le \partial(p,q) + \partial(q,r)$ (this is called the "triangle inequality").

<u>Main example</u>: X= $\mathbb{R}^n$ and $\partial^{Eucl}((x_1,\ldots,x_n), (y_1,\ldots,y_n)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + .. + (y_n - x_n)^2}$ the usual Euclidean distance. Also, for X and subset of $\mathbb{R}^n$, the restriction of ∂^{Eucl} to pairs in X defines a metric space.

<u>2 very interesting examples</u>.
(1) Let [a,b] be any interval in $\mathbb{R}$. Let $C([a,b],\mathbb{R})$ denote the set of all real-valued continuous functions y(t) defined on [a,b]. The distance is defined to be $\partial(y_1, y_2) =$ maximum value of $|y_2(t) - y_1(t)|$ on [a,b]. By the maximum principle, this exists and is finite. If $\partial(y_1,y_2) = 0$, then for all t, $|y_2(t) - y_1(t)| \le 0$, i.e. $y_2(t) = y_1(t)$. So if $\partial(y_2,y_1) = 0$, then $y_2 = y_1$. Symmetry of ∂ is obvious.

<u>Exercise</u> (not to be handed in): Check that $\partial(y_1,y_2)$ satisfies the triangle inequality.

(2) A slight variation of the last example is the one x we are interested in: $B_{\|\bullet\|}(y_0, b)$ is defined to be the set of all real-valued continuous functions y(t) on $[t_0, t_0+c]$ such that for all t, $|y(t) - y_0| \le b$.

In other words, the set of continuous functions whose graph lies in the rectangle $[t_0, t_0 + c] \times [y_0 - b, y_0 + b]$. The distance function is the same as in the previous example.

<u>Def'n</u>: Let (X,∂) be a metric space. <u>A contraction mapping on (X,d)</u> is a mapping T defined on X and taking values in X such that for every pair $p \neq q$ p,q of elements in X, $\partial(T(q), T(p)) < \partial(q, p)$. Let $o < \varepsilon < 1$ be a real number. The mapping T is an <u>ε-contraction mapping</u> if for all p,q, we have $\partial(T(q), T(p)) \le \varepsilon \bullet \partial(q, p)$.

Given a mapping T defined on X and taking values in X, a <u>fixed point</u> is an element p in X such that $T(p) = p$, i.e. p is <u>fixed</u> by T. Many important theorems in mathematics are descriptions of fixed points of certain mappings. One of the simplest theorems is the "contraction mapping fixed point thm".

<u>Thm</u> [Contraction mapping fixed point thm, part I]: Let T be a contraction mapping on (X,∂). There is at most 1 fixed point of T.

<u>Pf</u>: If T has no fixed points, the theorem is vacuously true. Thus suppose there exists a fixed point p. Let q be any point other than p. Then $\partial(p, T(q)) = (T(p), T(q))$ (b/c $p = T(p)$)

$< \partial(p, q)$ (b/c T is a contraction mapping).

Since $\partial(p, T(q)) \neq \partial(p, q)$, in particular $T(q) \neq q$. So p is the unique fixed point.

The second half of this theorem will assert that there exists a fixed point of T if T is an ε-contraction mapping for some $0 < \varepsilon < 1$ and if X "has no holes".

5. The integral operator.

Let y(t) be a continuous function defined on $[t_0, t_0+c]$ whose graph is contained in R. Define z = T(y) to be the function defined on $[t_0, t_0+c]$ by

$$z(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds .$$

By the fundamental theorem of calculus, z(t) is differentiable and $z'(t) = f(t, y(t))$. In particular, z(t) is continuous.

Suppose that the graph of z is in $[t_0, t_0 + c] \times [y_0 - a, y_0 + a]$.

Then for every t, $|z(t) - y_0| \le \left| \int_{t_0}^{t} f(s, y(s))ds \right|$

$$\le \int_{t_0}^{t} \left| f(s, y(s)) \right| ds \le \int_{t_0}^{t} M\partial s = M(t - t_0) \le M \cdot c .$$

Because we chose $c < \frac{b}{M}$, $|z(t) - y_0| < M \cdot c < b$. Thus z(t) is again an element of $B_{\|\bullet\|}(y_0, b)$.

So the rule that associates to each y(t) the function z is a mapping T from $B_{\|\bullet\|}(y_0, b)$ to itself.

Moreover, for any 2 elements y_1, y_2, for every t,

$$|z_2(t) - z_1 t| = \left| \int_{t_0}^{t} f(s, y_2(s)) - f(s, y_1(s))\ ds \right|$$

$$\leq \int_{t_0}^{t} L . |y_2(s) - y_1(s)| ds \qquad \text{(b/c of the Lipschitz condition satisfied by f)}$$

$$\leq \int_{t_0}^{t} L \cdot \max\{|y_2 - y_1|\}\ ds = \mathrm{L} \cdot \partial(\mathrm{y}_2, \mathrm{y}_1) \cdot (\mathrm{t} - \mathrm{t}_0) \leq \mathrm{L} \cdot \partial(\mathrm{y}_2, \mathrm{y}_1) \cdot c$$

Because we chose $c \leq \frac{1}{2L}$, we conclude $|z_2(t) - z_1 t| \leq \frac{1}{2} \partial(y_2, y_1)$ for all t. Therefore $\partial(\mathrm{T(y2)}, \mathrm{T(y1)}) \leq \frac{1}{2} \partial(\mathrm{y2}, \mathrm{y1})$. By part 1 of the contraction mapping fixed point theorem, there is at most one fixed point of T.

6. Tying up loose ends.

<u>Thm</u>: Let R be a region in the plane, let f be a function on R, and let (t_0,y_0) be a point of R. Let I=(a,b) be an interval containing t_0, and let y_1,y_2 be differentiable functions on (a,b) whose graphs are contained in R, both of which solve the IVP

$$\begin{cases} \mathrm{y}' = \mathrm{f(t, y)} \\ \mathrm{y(t_0)} = \mathrm{y}_0 \end{cases}$$

If f and $\frac{\partial f}{\partial y}$ are continuous on R, then $\mathrm{y}_1 = \mathrm{y}_2$.

<u>Proof</u>: We prove that the restrictions of y_1 and y_2 to $[t_0,b]$ are equal. The proof that the restrictions to $[a,t_0]$ are equal is almost the same. If $y_2 \neq y_1$ on $[t_0,b]$, then there exits a largest number t_1 with $\mathrm{t}_0 \leq \mathrm{t}_1 < \mathrm{b}$ such that the restrictions of y_1,y_2 to $[t_0,t_1]$ are equal. After replacing t_0 by t_1 and y_0 by $\mathrm{y}_1(\mathrm{t}) = \mathrm{y}_2(\mathrm{t}_1)$, we may suppose that y_1 & y_2 are sol'ns and for all ε>0, there exists t with $t_0 \leq t_1 < t0 + \varepsilon$ such that $\mathrm{y}_2(\mathrm{t}) \neq \mathrm{y}_1(\mathrm{t})$. Let

$[\mathrm{t}_0 - \mathrm{a}, \mathrm{t}_0 + \mathrm{a}] \times [\mathrm{y}_0 - \mathrm{b}, \mathrm{y}_0 + \mathrm{b}]$ be a rectangle contained in R. Let M and L be as above. Because y_2 and y_1 are continuous, there exists c<$\min\left\{a, \frac{b}{M}, \frac{1}{2L}\right\}$ such that for the restrictions of y_1 and y_2 to $[t_0,t_0+c]$, the graph is contained in $[t_0,t_0+c]$x$[y_0-b,y_0+b]$. So y_1, y_2 are in the metric space $B_{\|\bullet\|}(y_0, b)$. By the fundamental thm of calculus,

$$\mathrm{y_i(t)} = \mathrm{y}_0 + \int_{t_0}^{t} y_i'(s) ds = y_0 + \int_{t_0}^{t} f(s, y_i(s) ds = \mathrm{T(y_i)(t)}.$$

So y_1 & y_2 are different fixed points of T. This contradicts the contraction mapping fixed point thm. The only possible conclusion is that $\mathrm{y}_1 = \mathrm{y}_2$