

**18.034 EXAM 1**  
**FEBRUARY 25, 2004**

Name: \_\_\_\_\_

Problem 1: \_\_\_\_\_ /25

Problem 2: \_\_\_\_\_ /25

Problem 3: \_\_\_\_\_ /20

Problem 4: \_\_\_\_\_ /15

Problem 5: \_\_\_\_\_ /15

Total: \_\_\_\_\_ /100

**Instructions:** *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

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*Date:* Spring 2004.

Name: \_\_\_\_\_

Problem 1: \_\_\_\_\_ /25

**Problem 1**(25 points) Contaminated fluid flows into a large container with volume  $V$  at a rate of  $a$  liters/sec. The concentration of the contaminant is a constant  $q$ . Denote by  $y(t)$  the total mass of contaminant in the container at time  $t$ . Chemical reactions neutralize the contaminant in the container at a rate  $by(t)$ , where  $b$  is a constant. Treated fluid flows out of the container at the rate of  $a$  liters/sec. The constants  $a$ ,  $q$  and  $V$  are positive, and  $b$  is nonnegative.

(a)(10 points) Find a first-order linear ODE in normal form that  $y(t)$  satisfies.

(b)(10 points) At time  $t = 0$ , the amount of contaminant in the container is  $y_0$ . Using an integrating factor, find  $y(t)$  for  $t > 0$ .

(c)(5 points) In the steady-state, what is the concentration of contaminant in the outflowing fluid, i.e. what is  $y(t)/V$ ? How does the steady-state concentration for  $b > 0$  compare to the steady-state concentration for  $b = 0$ ?

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Problem 2: \_\_\_\_\_ /25

**Problem 2**(25 points) Consider the first-order ODE,

$$(y - t)y' + (2t - y) = 0,$$

where  $y = y(t)$ . This ODE is exact.

(a)(10 points) Find a non-constant function  $H(t, y)$  that is constant on all solution curves.

(b)(10 points) Write down the equation of a general integral curve of this system. What part or parts of this integral curve are solution curves?

(c)(5 points) If the initial value  $y(0) = y_0$  is positive, how much time elapses before the solution of the IVP is undefined?

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Problem 3: \_\_\_\_\_ /10

**Problem 3**(20 points) The following first-order ODE in normal form is a *Bernoulli equation*,

$$y' = 2\frac{y}{t} - ty^2, \quad t > 0.$$

To simplify this ODE, substitute  $u = 1/y$ ,  $u' = -y'/y^2$ .

(a)(15 points) Rewrite the ODE in terms of  $t$  and  $u$ , solve the resulting ODE, and back-substitute to find  $y(t)$ .

(b)(5 points) Were any solutions lost by making the substitution? If so, say what they are.

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Problem 4: \_\_\_\_\_ /20

**Problem 4**(15 points) Consider the first-order ODE with initial condition,

$$\begin{cases} y' = y, \\ y(0) = 1 \end{cases}$$

**(a)**(5 points) For the first-approximation function, take  $y_0(t) = 1$ . Compute the first 2 Picard iterates,  $y_1(t)$  and  $y_2(t)$ .

**(b)**(10 points) Find a formula for the  $n^{\text{th}}$  Picard iterate,  $y_n(t)$ . Justify your answer.

**Extra credit**(5 points) Find the formula for the  $n^{\text{th}}$  Picard iterate associated to the IVP,

$$\begin{cases} y' = \cos(t), \\ y(0) = 0, \end{cases}$$

For first-approximation function, take  $y_0(t) = 0$ .

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Problem 5: \_\_\_\_\_ /15

**Problem 5**(15 points) Consider the first-order, autonomous ODE,

$$y' = y^3 - y.$$

**(a)**(10 points) Find all equilibrium solutions, sketch the state line, and sketch several solutions including all equilibrium solutions, and at least one solution between any 2 equilibrium solutions.

**(b)**(5 points) For each equilibrium solution, state whether it is stable, unstable or neither.