18.034 SOLUTIONS TO PROBLEM SET 9

Due date: Friday, April 30 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(1)(5 points) p. 403, Problem 5

Solution: The trace is $\text{Trace}(A) = 6$, and the determinant is $\det(A) = 2{\cdot}4-(-1)1 = 9$. Therefore the characteristic polynomial is,

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Therefore there is one eigenvalue $\lambda = 3$ with multiplicity 2. Because the matrix is not diagonal, the eigenspace is deficient. Therefore there is a generalized eigenvector. The matrix $A - 3I$ is,

$$A - 3I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

A generalized eigenvector is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the corresponding vector $\mathbf{v}_1 = (A - 3I)\mathbf{v}_2$ is,

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

The change-of-basis matrix is,

$$U = [\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $AU = UD$ where D is the matrix in Jordan canonical form,

$$D = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

Therefore,

$$\exp(tA)U = U\exp(tD).$$

The solution space of the system,

$$\mathbf{z}'(t) = D\mathbf{z}(t),$$

has basis,

$$\mathbf{z}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t},\ \mathbf{z}_2(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} e^{3t}.$$

Therefore the basic solution matrix is,

$$Z(t) = [\mathbf{z}_1(t)|\mathbf{z}_2(t)] = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{3t}.$$

Of course $Z(0) = I$, therefore $\exp(tD) = Z(t)$. Therefore,

$$\exp(tA) = U\exp(tD)U^{-1}.$$

By Cramer's rule,

$$U^{-1} = (1/\det(U)) \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Therefore,

$$\exp(tA) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{3t} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Simplifying, this gives,

$$\exp(tA) = \begin{bmatrix} -t+1 & t \\ -t & t+1 \end{bmatrix} e^{3t} = \begin{bmatrix} (-t+1)e^{3t} & te^{3t} \\ -te^{3t} & (t+1)e^{3t} \end{bmatrix}.$$

To double-check, observe the derivative is,

$$\frac{d}{dt}\exp(tA) = \begin{bmatrix} (-3t+2)e^{3t} & (3t+1)e^{3t} \\ (-3t-1)e^{3t} & (3t+4)e^{3t} \end{bmatrix}.$$

This is the same as,

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} (-t+1)e^{3t} & te^{3t} \\ -te^{3t} & (t+1)e^{3t} \end{bmatrix} = A\exp(tA).$$

And $\exp(0A) = I$.

(2)(5 points) p. 403, Problem 11

Solution: This is one of the very few situations where the power series definition of the matrix exponential is useful. Observe that,

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix},$$

and,

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} = 0.$$

Therefore $A^n = 0$ for $n \geq 3$. So the power series,

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n,$$

reduces to,

$$\exp(tA) = I + t + \frac{1}{2}t^2A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 4t^2+3t & 4t & 1 \end{bmatrix}.$$

Therefore the solution of the IVP,

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t), \\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{cases}$$

has solution,

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 4t^2+3t & 4t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2t+2 \\ 4t^2+8t+3 \end{bmatrix}.$$

As a double-check, observe the derivative is,

$$\mathbf{x}'(t) = \begin{bmatrix} 0 \\ 2 \\ 8t+11 \end{bmatrix}.$$

This is the same as,

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2t+2 \\ 4t^2+8t+3 \end{bmatrix} = A\mathbf{x}(t).$$

And $\mathbf{x}(0) = [1\ 2\ 3]^{\mathrm{T}}$.

(3)(10 points) p. 403, Problem 15

Solution: The trace is $\mathrm{Trace}(A) = 0$, and the determinant is $\det(A) = 2(-2) - (-1)3 = -1$. Therefore the characteristic polynomial is,

$$p_A(\lambda) = \lambda^2 - \mathrm{Trace}(A)\lambda + \det(A) = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

Therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$, each with multiplicity 1. For $\lambda_1 = -1$, the matrix $A - \lambda_1 I$ is,

$$A + I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}.$$

An eigenvector is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

For $\lambda_2 = 1$, the matrix $A - \lambda_2 I$ is,

$$A - I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}.$$

An eigenvector is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The change-of-basis matrix is,

$$U = [\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Then $AU = UD$ where D is the diagonal matrix,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\exp(tD) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix}.$$

And $\exp(tA) = U\exp(tD)U^{-1}$. By Cramer's rule,

$$U^{-1} = (1/\det(U)) \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}.$$

Therefore,

$$\exp(tA) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}.$$

Simplifying, this gives,

$$\exp(tA) = \frac{1}{2} \begin{bmatrix} -e^{-t} + 3e^t & e^{-t} - e^t \\ -3e^{-t} + 3e^t & 3e^{-t} - e^t \end{bmatrix}.$$

Therefore,

$$\exp(tA)\mathbf{x}^0 = \frac{1}{2}\begin{bmatrix} -e^{-t}+3e^t & e^{-t}-e^t \\ -3e^{-t}+3e^t & 3e^{-t}-e^t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t}+e^t \\ 3e^{-t}+e^t \end{bmatrix}.$$

Also,

$$\exp(-sA)\mathbf{F}(s) = \frac{1}{2}\begin{bmatrix} -e^{s}+3e^{-s} & e^{s}-e^{-s} \\ -3e^{s}+3e^{-s} & 3e^{s}-e^{-s} \end{bmatrix}\begin{bmatrix} 3e^s \\ s \end{bmatrix}.$$

Simplifying, this gives,

$$\exp(-sA)\mathbf{F}(s) = \frac{1}{2}\begin{bmatrix} -3e^{2s}+9+se^s-se^{-s} \\ -9e^{2s}+9+3se^s-se^{-s} \end{bmatrix}.$$

So the final answer is,

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} e^{-t}+e^t \\ 3e^{-t}+e^t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -e^{-t}+3e^t & e^{-t}-e^t \\ -3e^{-t}+3e^t & 3e^{-t}-e^t \end{bmatrix}\int_0^t \frac{1}{2}\begin{bmatrix} -3e^{2s}+9+se^s-se^{-s} \\ -9e^{2s}+9+3se^s-se^{-s} \end{bmatrix} ds.$$

The following was *NOT ASKED FOR IN THE EXERCISE.* However, I can't resist mentioning how much simpler the solution is if one does not use the matrix exponential. Since we have the transition matrix, the simplest solution is to change the basis,

$$\mathbf{x}(t) = U\mathbf{z}(t).$$

Then,

$$\mathbf{z}^0 = U^{-1}\mathbf{x}^0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and,

$$\mathbf{G}(t) = U^{-1}\mathbf{F}(t) = \frac{1}{2}\begin{bmatrix} -3 \\ 9 \end{bmatrix} e^t + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} t.$$

The original inhomogeneous IVP is equivalent to the IVP,

$$\begin{cases} \mathbf{z}'(t) = D\mathbf{z}(t) + \mathbf{G}(t) \\ \mathbf{z}(0) = \mathbf{z}^0 \end{cases}$$

Now this is a set of two uncoupled inhomogeneous 1^{st} order linear IVPs,

$$\begin{cases} z_1' + z_1 = -\frac{3}{2}e^t + \frac{1}{2}t, & z_1(0) = \frac{1}{2} \\ z_2' - z_2 = \frac{9}{2}e^2 - \frac{1}{2}t, & z_2(0) = \frac{1}{2} \end{cases}$$

By the method of undetermined coefficients, particular solutions of the inhomogeneous ODEs (but not of the initial conditions!) are,

$$\begin{cases} z_1(t) = -\frac{3}{4}e^t + \frac{1}{2}t - \frac{1}{2} \\ z_2(t) = \frac{9}{2}te^t + \frac{1}{2}t + \frac{1}{2} \end{cases}$$

The general solution of the homogeneous equation is $z_1(t) = C_1e^{-t}$, $z_2(t) = C_2e^t$. Therefore the solution of the system of IVPs is,

$$\begin{cases} z_1(t) = -\frac{3}{4}e^t + \frac{1}{2}t - \frac{1}{2} + \frac{7}{4}e^{-t} \\ z_2(t) = \frac{9}{2}te^t + \frac{1}{2}t + \frac{1}{2} \end{cases}$$

In vector form,

$$\mathbf{z}(t) = \frac{1}{4}\begin{bmatrix} -3e^t + 2t - 2 + 7e^{-t} \\ 18te^t + 2t + 2 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}(t) = U\mathbf{z}(t) = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -3e^t + 2t - 2 + 7e^{-t} \\ 18te^t + 2t + 2 \end{bmatrix}.$$

Simplifying, this gives,

$$\mathbf{x}(t) = \frac{1}{4}\begin{bmatrix} 18te^t - 3e^t + 4t + 7e^{-t} \\ 18te^t - 9e^t + 8t - 4 + 21e^{-t} \end{bmatrix}.$$

(4)(10 points) p. 403, Problem 25

Solution: First of all,

$$\exp(tA) = I + tA = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

and,

$$\exp(tB) = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$\exp(tA)\exp(tB) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & t \\ 0 & 1 \end{bmatrix}.$$

Also,

$$\exp(tB)\exp(tA) = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & 1 \end{bmatrix}.$$

Therefore $\exp(tA)\exp(tB)$ does not equal $\exp(tB)\exp(tA)$.

Also,

$$A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Because this is an upper triangular matrix, $p_{A+B}(\lambda) = (\lambda - 1)(\lambda - 0)$. So the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$. The eigenvector for $\lambda_1 = 0$ is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The eigenvector for $\lambda_2 = 1$ is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So the change-of-basis matrix is,

$$U = [\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

And $(A + B)U = UD$ where D is the diagonal matrix,

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Of course,

$$\exp(tD) = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}.$$

By Cramer's rule,

$$U^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$\exp(t(A + B)) = U\exp(tD)U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Simplifying, this is,

$$\exp(t(A+B)) = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore $\exp(t(A+B))$ equals neither $\exp(tA)\exp(tB)$ nor $\exp(tB)\exp(tA)$.

(5)(10 points) p. 420, Problem 4

Solution: Consider the complexified ODE,

$$\widetilde{\mathbf{x}}'(t) - \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \widetilde{\mathbf{x}}(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{i\omega t}.$$

First consider the case that $\omega \neq -2$. By the method of undetermined coefficients, we guess that a particular solution is,

$$\widetilde{\mathbf{x}}_p(t) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{i\omega t}.$$

Plugging in gives the linear equations,

$$\begin{bmatrix} i\omega C_1 - 2C_2 \\ 2C_1 + i\omega C_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{i\omega t}.$$

The unique solution of this system of linear equations is,

$$\begin{cases} C_1 & = & -i/(2+\omega), \\ C_2 & = & -1/(2+\omega) \end{cases}$$

Therefore the particular solution of the complexified ODE is,

$$\widetilde{\mathbf{x}}_p(t) = \frac{-1}{2+\omega} \begin{bmatrix} i \\ 1 \end{bmatrix} e^{i\omega t}.$$

So the real part is,

$$\mathbf{x}_p(t) = \frac{1}{2+\omega} \begin{bmatrix} \sin(\omega t) \\ -\cos(\omega t) \end{bmatrix}.$$

Since the general solution of the homogeneous equation is,

$$D_1 \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + D_2 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix},$$

the solution of the IVP is,

$$\mathbf{x}(t) = \frac{1}{2+\omega} \begin{bmatrix} \sin(\omega t) \\ -\cos(\omega t) \end{bmatrix} + a \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + \left(b + \frac{1}{2+\omega} \right) \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.$$

Next suppose that $\omega = -2$. Then, again by the method of undetermined coefficients, we guess that a particular solution is,

$$\widetilde{\mathbf{x}}_p(t) = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} te^{-2it} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{-2it}.$$

Plugging in and solving the system of linear equations, one particular solution is,

$$\widetilde{\mathbf{x}}_p(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} te^{-2it}.$$

Therefore the real part is,

$$\mathbf{x}_p(t) = \begin{bmatrix} t\cos(2t) \\ -t\sin(2t) \end{bmatrix}.$$

And the solution of the initial value problem is,

$$\mathbf{x}(t) = \begin{bmatrix} t\cos(2t) \\ -t\sin(2t) \end{bmatrix} + a \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + b \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}.$$

The response of the system is unbounded iff $\omega = -2$.

(6)(10 points) p. 420, Problem 11

Solution: The goal is to prove Theorem 6.8.5 (which, incidentally, is essentially equivalent to our Green's kernel solution). Let $\mathbf{c}(t)$ be a continuously differentiable vector-valued function. Consider the continuously differentiable vector-value function,

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{c}(t).$$

This is a solution of the IVP iff

$$\begin{cases} \Phi'(t, t_0)\mathbf{c}(t) + \Phi(t, t_0)\mathbf{c}'(t) = A(t)\Phi(t, t_0)\mathbf{c}(t) + \mathbf{F}(t), \\ \Phi(t_0, t_0)\mathbf{c}(t_0) = 0 \end{cases}$$

By hypothesis, $\Phi'(t, t_0) = A(t)\Phi(t, t_0)$ and $\Phi(t_0, t_0) = I_n$. Therefore $\mathbf{x}(t)$ is a solution of the IVP iff,

$$\begin{cases} \Phi(t, t_0)\mathbf{c}'(t) = \mathbf{F}(t), \\ \mathbf{c}(t_0) = 0 \end{cases}$$

By the last part of Theorem 6.8.4, $\Phi(t_0, t)\Phi(t, t_0) = I_n$. Therefore $\mathbf{x}(t)$ is a solution of the IVP iff,

$$\begin{cases} \mathbf{c}'(t) = \Phi(t_0, t)\mathbf{F}(t), \\ \mathbf{c}(t_0) = 0 \end{cases}$$

By the Fundamental Theorem of Calculus, there is a unique solution of this IVP, and it is given by,

$$\mathbf{c}(t) = \int_{t_0}^{t} \Phi(t_0, s)\mathbf{F}(s)ds.$$

Therefore, the unique solution of the IVP is given by,

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{c}(t) = \Phi(t, t_0)\int_{t_0}^{t} \Phi(t_0, s)\mathbf{F}(s)ds.$$

The second part of Theorem 6.8.5 follows easily from the first part.