Recitation 15, April 1, 2010

Fourier Series: Harmonic response

1. Let $f(t)$ denote the even function $f(t)$ which is periodic of period 2π and such that $f(t) = |t|$ for $-\pi < t < \pi$. Graph $f(t)$. In lecture we found that the Fourier series of $f(t)$ is given by

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Now we want to alter $f(t)$ to produce a function $g(t)$ whose graph is the same as that of $f(t)$ but is compressed (or expanded) horizontally so that the circular frequency is ω. What is the formula for $g(t)$ in terms of $f(t)$? Use the Fourier series for $f(t)$ and a substitution to find the Fourier series for the function $g(t)$.

The graph of f is a triangular wave, with minimum 0 at zero, and maximum π.

$g(t) = f(\omega t)$. The circular frequency of f is $2\pi/2\pi = 1$, so the circular frequency of $g(t)$ is ω times that. The Fourier series for g has the same coefficients as that for f, but the frequencies are multiplied by ω:

$$g(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(\omega t) + \frac{\cos(3\omega t)}{3^2} + \frac{\cos(5\omega t)}{5^2} + \cdots\right)$$

2. Next drive the harmonic oscillator with the function $f(t)$ from **(1)**:
$\ddot{x} + \omega_n^2 x = f(t)$. Find a periodic solution, when one exists, as a Fourier series.

By superposition, the driven oscillator experiences resonance if and only if ω_n is the circular frequency of a term in the Fourier expansion of f with nonzero coefficient. In our case, we get periodic solutions for all $\omega_n \neq 0, \pm 1, \pm 3, \pm 5, \pm 7, \ldots$. Under this condition, by the first formula in the second box, we know when k is odd, $\ddot{x} + \omega_n^2 x = \frac{\cos(k\omega t)}{k^2}$ has a periodic solution $x(t) = \frac{\cos(k\omega t)}{k^2(\omega_n^2 - k^2)}$; furthermore, $\ddot{x} + \omega_n^2 x = \frac{\pi}{2}$ also has a periodic solution which is $x \equiv \frac{\pi}{2\omega_n^2}$. So, again by superposition, the original equation has the following periodic solution:

$$x_p(t) = \frac{\pi}{2\omega_n^2} - \frac{4}{\pi}\left(\frac{\cos t}{\omega_n^2 - 1} + \frac{\cos 3t}{9(\omega_n^2 - 9)} + \frac{\cos 5t}{25(\omega_n^2 - 25)} + \ldots\right),$$

and $x_p(t)$ is given by its Fourier series.

3. Now drive the harmonic oscillator with the function $g(t)$ from **(1)** of circular frequency ω: $\ddot{x} + \omega_n^2 x = g(t)$. Again, find a periodic solution, when one exists.

Again we take superposition of termwise solutions, but this time, k is replaced by $k\omega$ every time it appears. Therefore, when $\omega_n \neq 0, \pm\omega, \pm 3\omega, \pm 5\omega, \ldots$, there exists a periodic solution, and it's given by its Fourier series as the following:

$$x_p = \frac{\pi}{2\omega_n^2} - \frac{4}{\pi}\left(\frac{\cos \omega t}{\omega_n^2 - \omega^2} + \frac{\cos(3\omega t)}{9(\omega_n^2 - 9\omega^2)} + \frac{\cos(5\omega t)}{25(\omega_n^2 - 25\omega^2)} + \ldots\right).$$

4. Suppose that ω is fixed, but we can vary ω_n. We may have a radio receiver, for example, and we want to pick up (amplify) radio signals at or near a certain circular frequency, so we set the capacitance so that the natural circular frequency of the circuit is ω_n. At what values of ω_n does the harmonic oscillator fail to have a periodic system response? (This is resonance.) Describe the system response when ω_n is just larger or just smaller than one of those values?

The harmonic oscillator fails to have a periodic system response when ω_n is an odd multiple of ω or 0. When ω_n is just larger or just smaller than one of those values, the gain for that frequency component is large, and the Fourier series for the solution has a large coefficient for that frequency.

5. Are there frequencies at which there is more than one periodic solution?

Yes. In general, when $\omega_n \neq 0$, any nonzero solution to the homogeneous equation $\ddot{x} + \omega_n^2 x = 0$ has period $\frac{2\pi}{\omega_n}$, and we can add such solutions to periodic solutions to the inhomogeneous equation, and still get solutions to the inhomogeneous equation. Furthermore, if there exits a pair of integers (a, b) such that $a\omega_n = b\omega$ and b is not an odd multiple of a, then the resulting function is also periodic with a period equal to $\frac{2\pi b}{\omega_n}$.

18.03 Differential Equations
Spring 2010