Recitation 11, March 11, 2010

Frequency Response

This project will be much more meaningful if it is accompanied by the Mathlet `Amplitude and Phase: Second order, IV` (available at http://math.mit.edu/daimp/). This illustrates the second order mass/spring/dashpot system driven by a force F_{ext} acting directly on the mass: $m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$. So the input signal is F_{ext} and the system response is x. We're interested in sinusoidal input signal, $F_{\text{ext}}(t) = A\cos(\omega t)$, and in the steady state, sinusoidal system response, $x_p(t) = gA\cos(\omega t - \phi)$. Here g is the gain of the system and ϕ is the phase lag. Both depend upon ω, and we will consider how. We might as well take $A = 1$, so the amplitude of the system response equals the gain.

Take $m = 1$, $b = \frac{1}{4}$, and $k = 2$.

1. Compute the complex gain $H(\omega)$ of this system. (This means: make the complex replacement $F_{\text{cx}} = e^{i\omega t}$, and express the exponential system response z_p as a complex multiple of F_{cx}: $z_p = H(\omega)F_{\text{cx}}$.)

Set $F_{\text{cx}} = e^{i\omega t}$. The complex replacement of the equation is $\ddot{z} + \frac{1}{4}\dot{z} + 2z = e^{i\omega t}$, with the characteristic polynomial $p(s) = s^2 + \frac{1}{4}s + 2$. $p(i\omega) = -\omega^2 + \frac{\omega}{4}i + 2 \neq 0$, so by the ERF, $z_p = e^{i\omega t}/p(i\omega) = F_{\text{cx}}/p(i\omega)$, and $H(\omega) = z_p/F_{\text{cx}} = 1/p(i\omega) = \frac{2-\omega^2-\omega i/4}{(2-\omega^2)^2+(\omega/4)^2} = \frac{2-\omega^2-\omega i/4}{\omega^4 - \frac{63}{16}\omega^2 + 4}$.

2. Write down the expression for the gain $g(\omega) = |H(\omega)|$. What is the amplitude of the system response when $\omega = 1$? (You can check your answer using the applet.)

$g(\omega) = |H(\omega)| = \frac{1}{|p(i\omega)|} = \left(\omega^4 - \frac{63}{16}\omega^2 + 4\right)^{-\frac{1}{2}}$. Since $A = 1$, $g(\omega)$ equals the amplitude of the system response. So when $\omega = 1$, the amplitude of the system response is $g(1) = \frac{4}{\sqrt{17}}$.

3. What is the resonant circular frequency ω_r? (Hint: minimize the square of the denominator.)

$g(\omega)$ will be maximized at the resonant circular frequency ω_r. Equivalently, $g(\omega)^{-2} = \omega^4 - \frac{63}{16}\omega^2 + 4$ will be minimized at ω_r. To find out ω_r, we start with the critical points of $\omega^4 - \frac{63}{16}\omega^2 + 4$. Taking the first deriviative in ω and setting it zero leads to $4\omega^3 - \frac{63}{8}\omega = 0$, which implies $\omega = 0$ or $\omega^2 = \frac{63}{32}$, i.e., $\omega = \pm\frac{3\sqrt{14}}{8}$. We want to take the positive value, so $\omega_r = \frac{3\sqrt{14}}{8}$. To verify this gives a minimum, we use the second derivative test. $\left(\omega^4 - \frac{63}{16}\omega^2 + 4\right)'' = 12\omega^2 - \frac{63}{8}$, which is positive at $\omega_r = \frac{3\sqrt{14}}{8}$, so $g(\omega_r)^{-2}$ is a minimum.

4. It appears that the phase lag is approximately $\frac{\pi}{2}$ at the resonant circular frequency. Is that correct? That is, at what frequency is the phase lag equal to one quarter cycle?

The response of the original system is $x_p = Re(z_p) = |H(\omega)|\cos(\omega t + \arg(H(\omega)))$. So the phase lag ϕ is equal to $-\arg(H(\omega)) = \arg(p(i\omega))$. As seen in question 1, $H(\omega) = \frac{2-\omega^2-\omega i/4}{\omega^4 - \frac{63}{16}\omega^2+4}$, so $\cos\phi = \cos(\arg(p(i\omega))) = \frac{2-\omega^2}{\sqrt{\omega^4-\frac{63}{16}\omega^2+4}}$, and $\sin\phi = \sin(\arg(p(i\omega))) = \frac{\omega/4}{\sqrt{\omega^4-\frac{63}{16}\omega^2+4}}$. When $\omega = \omega_r = \frac{3\sqrt{14}}{8}$, $\cos\phi = \frac{1}{\sqrt{127}} \approx 0.089$ and $\sin\phi = \frac{3\sqrt{14}}{\sqrt{127}} \approx 0.996$. ϕ should be very close to $\pi/2$, since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

5. At what circular frequency is the phase lag equal to $\frac{\pi}{4}$? How about $\frac{3\pi}{4}$?

As seen in question 4, the phase lag ϕ has the property that $\cos\phi = \cos(\arg p(i\omega))) = \frac{2-\omega^2}{\sqrt{\omega^4-\frac{63}{16}\omega^2+4}}$, and $\sin\phi = \sin(\arg(p(i\omega))) = \frac{\omega/4}{\sqrt{\omega^4-\frac{63}{16}\omega^2+4}}$. When $\phi = \frac{\pi}{4}$, $\cos\phi = \sin\phi = \frac{\sqrt{2}}{2}$. It implies $2-\omega^2 = \omega/4$, i.e., $\omega = \frac{-1\pm\sqrt{129}}{8}$. We adopt the positive value, since $\omega/4 = (\sin\phi)\sqrt{\omega^4 - \frac{63}{16}\omega^2+4} = \frac{\sqrt{2}}{2}\sqrt{\omega^4 - \frac{63}{16}\omega^2+4} \geq 0$, so $\omega = \frac{-1+\sqrt{129}}{8}$. When $\phi = \frac{3\pi}{4}$, $-\cos\phi = \sin\phi = \frac{\sqrt{2}}{2}$. It implies $2-\omega^2 = -\omega/4$, i.e., $\omega = \frac{1\pm\sqrt{129}}{8}$. Again we adopt the positive value for the same reason as above, so $\omega = \frac{1+\sqrt{129}}{8}$.

6. New project: Find a solution of $\ddot{x} + 3\dot{x} + 2x = te^{-t}$.

We use the variation of parameter method. Suppose a solution is given in the form of $x(t) = u(t)e^{-t}$ for some function $u(t)$, then $\dot{x} = \dot{u}e^{-t} - ue^{-t}$ and $\ddot{x} = \ddot{u}e^{-t} - 2\dot{u}e^{-t} + ue^{-t}$. Plugging into the equation leads to $e^{-t}(\ddot{u}+\dot{u}) = te^{-t}$. Cancelling off e^{-t} from both sides, we get $\ddot{u}+\dot{u} = t$. To solve this equation for u, we use the undetermined coefficient method. However, the corresponding charateristic polynomial $p(s) = s^2 + s$ has zero as its constant term. So set $w = \dot{u}$, then the equation can be rewritten as $\dot{w} + w = t$. This can be solved and one solution is $w = t-1$, and hence $\dot{u} = t-1$, and one solution for u is $u = \frac{t^2}{2} - t$. Back to the original equation, one solution is given by $x = (\frac{t^2}{2} - t)e^{-t}$.

18.03 Differential Equations
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