18.03 Class 21, March 29

Fourier series II

[1] Review
[2] Square wave
[3] Piecewise continuity
[4] Tricks

[1] Recall from before break: A function f(t) is periodic of period 2L
if f(t+2L) = f(t) .

Theorem: Any decent periodic function f(t) of period 2pi has
can be written in exactly one way as a *Fourier series*:

f(t) = a_0/2 + a_1 cos(t) + a_2 cos(2t) + ...
+ b_1 sin(t) + b_2 sin(2t) + ...

If the need arises, the "Fourier coefficients" can be computed as
integrals:

a_n = (1/pi) integral_{-pi}^{pi} f(t) cos(nt) dt , n geq 0
b_n = (1/pi) integral_{-pi}^{pi} f(t) sin(nt) dt , n > 0

[2] Squarewave: A basic example is given by the "standard squarewave,"
which I denote by sq(t) : it has period 2pi and

sq(t) = 1 for 0 < t < pi
= -1 for -pi < t < 0
= 0 for t = 0 , t = pi

This is a standard building block for all sorts of "on/off" periodic
signals.

It's odd, so a_n = integral_{-¥pi}^pi odd . even dt = 0 for all n .

If f(t) is an odd function of period 2pi, we can simplify the
integral for bn a little bit. The integrand f(t) sin(nt) is even,
so the integral is twice the integral from 0 to pi:

bn = (2/pi) integral_0^pi f(t) sin(nt) dt

Similarly, if f(t) is even then

an = (2/pi) integral_0^pi f(t) cos(nt) dt

In our case this is particularly convenient, since sq(t) itself needs
different definitions depending on the sign of t. We have:

bn = (2/pi) integral_0^pi sin(nt) dt

= (2/pi) [ - cos(nt) / n ]_0^pi

= (2/pi n) [ - cos(n pi) - (-1) ]

$$= (2/\pi n) [ 1 - \cos(n \pi) ]$$

This depends upon $n$ :

| n | cos(n pi) | 1 - cos(n pi) |
|---|---|---|
| 1 | -1 | 2 |
| 2 | 1 | 0 |
| 3 | -1 | 2 |

and so on. Thus: $b_n = 0$ for $n$ even
$= 4\pi/n$ for $n$ odd and

$$sq(t) = (4/\pi) [ \sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + ... ]$$

This is the Fourier series for the standard squarewave.

I used the Mathlet FourierCoefficients to illustrate this. Actually, I built up the function

$$(\pi/4)\, sq(t) = \sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + .... \quad (**)$$

and observed the fit.

[3] What is "decent"?

This is quite amazing: the entire function is recovered from a *discrete* sequence of slider settings. They record the strength of the harmonics above the fundamental tone. The sequence of Fourier coefficients is a "transform" of the function, one which only applies (in this form at least) to periodic functions. We'll see another example of a transform later, the Laplace transform.

Let's be more precise about decency. First, a function is *piecewise continuous* if it is broken into continuous segments and such that at each point $t = a$ of discontinuity,

$$f(a-) = \lim_{t \to a \text{ from below}} f(t) \quad \text{and}$$

$$f(a+) = \lim_{t \to a \text{ from above}} f(t)$$

exist. They exist at points $t = a$ where $f(t)$ is continuous, too, and there they are equal. So $f(t) = 1/t$ is NOT piecewise continuous, but $sq(t)$ is .

A function is "decent" if it is piecewise continuous and is such that at each point of discontinuity, $t = a$ , the value at $a$ is the average of the left and right limits:

$$f(a) = (1/2) (f(a+) + f(a-))$$

So the square wave is decent, and any continuous function is decent.

Addendum to the theorem:

At points of discontinuity, the Fourier series can't make up its mind, so it converges to the average of f(a+) and f(a-) .

For example, evaluate the Fourier series for sq(t) at t = pi/2:

```
    sin( pi/2) = +1
    sin(3pi/2) = -1
    sin(5pi/2) = +1
          ...                             so

    1 = (4/pi) (1 - 1/3 + 1/5 - 1/7 + ...)    or

    1 - 1/3 + 1/5 - 1/7 + ...  =  pi/4
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Did you know this? It's due to Newton and Leibnitz.

[4] Tricks: Any way to get an expression (*) will give the same answer!

Example [trig id]: cos(t - pi/4) .

How to write it like (*) ? Well, there's a trig identity we can use:

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    a cos(t) + b sin(t) = A cos(t - phi)
    if  (a,b)  has polar coord's  (A,phi)

    a = A cos(phi),  b = A sin(phi) :
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For us, A = 1, phi = pi/4, so a = b = 1/sqrt(2) and

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    cos(t - pi/4) = (1/sqrt(2)) cos(t) + (1/sqrt(2) sin(t) .
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That's it: that's the Fourier series. This means a1 = b1 = sqrt(2) and all the others are zero.

Example [linear combinations]:

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    1 + 2 sq(t) = 1 + (8/pi) ( sin(t) + (1/3) sin(3t) + ... )
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Example [shifts]: f(t) = sq(t + pi/2)

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     =  (1/2) (4/pi) (sin(t + pi/2) + (1/3) sin(3(t + pi/2)) + ... )
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sin(theta + pi/2) = cos(theta) , sin(theta - pi/2) = - cos(theta) so

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    f(t)  =  (4/pi) (cos(t) - (1/3) cos(3t) + (1/5) cos(5t) - ... )
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Example [Stretching]: What about functions of other periods?
Suppose g(x) has period 2L .

Building blocks: cos(n(pi/L)x)
and sin(n(pi/L)x) are periodic of period 2L .

Then the Fourier series for g(x) is:

$$g(x) = a_0/2 + a_1 \cos((\pi/L)\, x) + a_2 \cos((2\pi/L)\, x) + \ldots$$
$$+ b_2 \sin((\pi/L)\, x) + b_2 \sin((2\pi/L)\, x) + \ldots$$

Example: sq((pi/2) x) has period 4 , not 2pi : L = 2. But we can still write (using the *substitution* t = (pi/2) x ) :

$$sq(2\pi x) = (4/\pi) ( \sin((\pi/2)x) + (1/3) \sin(3(\pi/2) x) + \ldots )$$

There are integral formulas as well. [Slide]