18.03 Class 16, March 10, 2010

Frequency response

[1] Variation of parameters
[2] Summary of complex gain
[3] Second order frequency response
[4: Supplement] First order frequency response
[5: Supplement] Other systems [not done in lecture]

[1] I promised on Monday to show you what you can do if A is not constant
in p(D)x = A e^{rt} .

Example: 3x" + 8x' + 6x = (t^2 + 1) e^{-t} .
Now B is not constant.

Try for a solution of the form x_p = u e^{-t} for some u .
This is what led us to the ERF; but now u is allowed to be nonconstant.
This is called *variation of parameters*.

        6]   x_p = u e^{-t}
        8]   x_p' = (u' - u) e^{-t}
        3]   x_p" = (u" - u' - u' + u) e^{-t}

  (t^2 + 1) e^{-t} = (3u" + 2u' + u) e^{-t}

Cancel the e^{-t} :   3u" + 2u' + u = t^2 + 1

This is solvable by undetermined coefficients. In fact, by an incredible
stroke of luck, we have already solved it!

        u_p = t^2 - 4t + 3

so   x_p = u_p e^{-t} = (t^2 - 4t + 3) e^{-t}

This method will always replace the given equation with another on
in which the right hand side is the same as before but without the e^{rt} .

[2] Complex gain summary:

General stuff (independent of order!) Input: y = A cos(omega t).

Complex input: y_cx = A e^{i omega t}

                 z_p = exponential system response to input y_cx

Complex gain: z_p = H(omega) y_cx

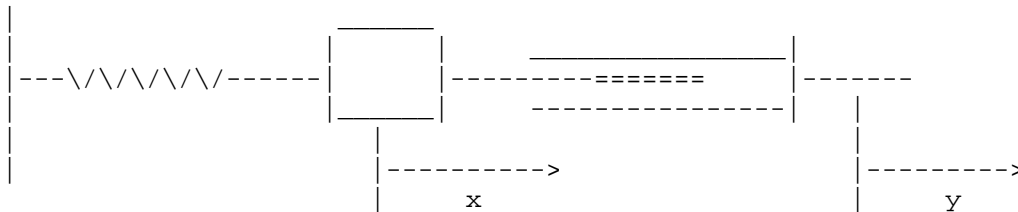Polar: H(omega) = |H(omega)| e^{- i phi}

                      z_p = |H(omega)| e^{- i phi} A e^{i omega t}

                          = |H(omega)| A e^{i(omega t - phi)}

Real part: x_p = |H(omega)| A cos(omega t - phi)

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Gain:                          g(omega) = |H(omega)|

Phase lag:                     phi(omega) = \phi = - Arg(H(omega))
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[3] I want to go back to the spring/mass/dashpot system and think
some more about what we can learn about the system response.

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   |                |_____|        ---------------|     |
   |                   |                              |
   |                |--------->                   |--------->
   |                |    x                        |    y
```

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        m x" + bx' + kx = by'            (*)
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Input signal: y = A cos(omega t)
System response: x

I demonstrated the Mathlet Amplitude and Phase: Second Order II.
It has m = 1 but we will work this out for general m .
It's interesting. What regularities can we see? Get responses from the class.

It seems that:

- When omega is small, g(omega) is small.

- The maximal amplitude of the system response is 1;
i.e. the maximal gain is 1. Write omega_r for that circular frequency;
it's the "resonant frequency."

- When omega is large, g(omega) is small.

and

- When omega is small, phi(omega) is about - pi/2 .

- phi(omega_r) = 0 (so the dashpot is "locked").

- When omega is large, phi(omega) is about + pi/2 .

Let's invoke the Bode plots: they support these observations. The hump
is called the "resonant peak." As long as b is not zero, this hump
does not go all the way to infinity; the maximal gain is finite.
I'll call the resonance we studied on Monday "true resonance."

From them you can see what happens when I fiddle with the system parameters.
Let's fix k but vary b .   It seems that:

- omega_r is independent of b .

- If I make b small, the resonant peak gets sharper.

- and the phase lag switches from - pi/2 to + pi/2 more abruptly.

This equation is the SAME as the equation modeling an AM radio receiver. Think of the input signal as an incoming radio wave, and the system response as the induced current in the receiver. The environment is full of radio waves of different frequencies. But only waves of frequency near to omega_r excite the receiver. This is "tuning." The sharper the spike, the better the reception.

Analysis: y = A cos(omega t)

Complex replacement: y_cx = A e^{i omega t}

         m z" + b z' + kz = b (y_cx)' = b A i omega e^{i omega t}

Steady state solution: z_p = b A i omega e^{i omega t} / p(i omega)

                         = (b i omega / p(i omega)) y_cx

              H(omega) = b i omega / p(i omega)

            p(i omega) = (k - m omega^2) + b i omega

          g(omega) = |H(omega)| = b omega / | p(i omega) |

                = b omega / sqrt ((k - m omega^2)^2 + b^2 omega^2)

When omega is small, this is small.

You can see that the denominator is always bigger than the numerator: the maximal gain is g = 1 , and this happens just when the extra term in the denominator, (k - m omega^2)^2 , vanishes: that is,

                    k = m omega_r^2

or   omega_r = sqrt(k/m)

which is the natural frequency of the system! This is a surprise!

                omega_r = omega_n

for this system. It is *independent of b* .

What is the phase lag at that frequency?

           H(omega_r) = b i omega / b i omega = 1

so the gain is 1 , as predicted, and the phase lag is 0 , as predicted.

All this is also clear from the Nyquist plot, which I invoked on the Mathlet. The Nyquist plot seems to be independent of the system parameters

(as it was in the first order case). It looks to be a circle of radius
1/2 centered at 1/2 . Almost exactly the same calculation we did before
shows that this is true.

[Here it is: Write x = b omega ,   a = k - m omega^2

and compute | (1/2) - ix/(a + ix) |

               = (1/2) | 1 - 2ix/(a + ix) |

               = (1/2) | (a + ix - 2ix) / (a + ix) |

               = (1/2) | a - ix | / | a + ix | = 1/2   ]

[4. Supplement] First order gain/phase lag analysis.

A bay on Cape Cod communicates with the ocean via a narrow channel.
The ocean experiences tides. What happens to the water level in the bay?

Linear model for this: y = ocean water level
                        x = bay water level

both with respect to the same mark, so that x = y means they are at
the same level.

      x < y ==> x' > 0
      x > y ==> x' < 0

so the linear model is   x' = k(y-x)   or   x' + kx = ky

k is the coupling constant.   This is just like Newtonian cooling.

Start with sinusoidal input: y = A cos(omega t) :

                       x' + ky = k A cos(omega t) .

I illustrated this using the Mathlet Amplitude and Phase: First Order.

Question: the cyan curve seems to cross the yellow curve right at the maxima
and minima. Can you explain that?
Work with your neighbor for a minute and I'll take proposals.

Ans: the yellow curve is x . At extrema, x' = 0 , which means that
ky = kx , or x = y .

Complex replacement: y_cx = A e^{i omega t}

                      z' + kz = k A e^{i omega t}

Steady state solution: z_p = k A e^{i omega t} / p(i omega)

                          = (k / (i omega + k)) A e^{i omega t}

In terms of complex gain: H(omega) = k / (i omega + k)

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                      g(omega) = k / sqrt(k^2 + omega^2)

                 tan(phi(omega)) = omega / k
```

Also phi is between 0 and pi/2, so

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                    phi(omega)  = arctan(omega/k)
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So now we have these formulas: the amplitude of the steady state solution of

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                    x' + ky = k A cos(omega t)
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is   g(omega) A = k A / sqrt(k^2 + omega^2)

and we understand the phase lag as well.

This computation gives equations for the Bode plots in the Mathlet.
A Bode plot is just a graph of the gain or phase lag against the
input frequency.

The "Nyquist plot" is the trajectory of the complex gain as a complex-valued
function of omega. In this case it looks like it sweeps out a semi-circle
which is independent of k . Let's check this:

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    1/2 - H(\omega)  =  1/2 - k/(i omega + k) =

                     =  ( (i omega + k) - 2k ) / 2 (i omega + k)

                     = (1/2) (( i omega - k ) / ( i omega + k )
```

Since |z/w| = |z|/|w| (because magnitudes multiply),

```
    1/2 - H(\omega)  =  (1/2) | i omega - k | / | i omega + k |
```

But k is the same distance from i omega   as - k is : so this ratio is 1.

[5. Supplement] Some other systems:

Drive through both spring and dashpot:

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          |---|      ==============---|      |
          |   |--------------         |      |
          |                           |_____|
          |                               |
          |----------->              |---------->
          |      y                   |      x
          |
```

Now   m x" + b x' + k x = k y + b y'

-- both position and velocity occur on the right hand side.

Still   Input signal = y
         Output   = x

This is illustrated by   Amplitude and Phase: Second Order III.

This is a standard model of an automobile suspension.
Now the gain can be well larger than 1, Nyquist plot depends heavily
on the system parameters.

18.03 Differential Equations
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