18.03 Problem Set 6: Part II Solutions

Part I points: 22. 8, **23.** 0, **24.** 8, **25.** 0.

22. (a) [4] From **II.21(f)**, $g(t) = \frac{4}{\pi}(\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) - \cdots)$. By Superposition III and the fact that $A\frac{\sin(\omega t)}{\omega_n^2 - \omega^2}$ is a solution to $\ddot{x} + \omega_n^2 x = A\sin(\omega t)$, we find that a solution to $\ddot{x} + \omega_n^2 x = g(t)$ is given by $x_p = \frac{4}{\pi}(\frac{\sin(t)}{\omega_n^2 - 1^2} - \frac{1}{3^2}\frac{\sin(3t)}{\omega_n^2 - 3^2} + \cdots)$, as long as ω_n is not an odd integer.

(b) [4] If ω_n is an odd integer there is no periodic solution.

(c) [4] $\omega_r = 1$. For ω just less than 1, the term $\frac{4}{\pi}\frac{\sin(t)}{\omega_n^2 - 1}$ dominates, and x_p is relatively close to this: This is *antiphase* with $\sin(t)$ and has large amplitude. When ω_n is just greater than 1, the same term occurs and dominates but now is a positive multiple of $\sin(t)$, so the system response is *in phase* with the input.

(d) [4] This is a tricky question. When ω_n is not an odd integer, the solution x_p above is periodic of period 2π. The general solution of the homogeneous equation is $a\cos(\omega_n t) + b\sin(\omega_n t)$, which is periodic of period $\frac{2\pi}{\omega_n}$. The sum is periodic if some multiple of 2π is equal to some multiple of $\frac{2\pi}{\omega_n}$, and this happens when ω_n is a rational number (but not an odd integer).

(e) [4] Yes. [They are periodic of period 2π if ω_n is an even integer.]

23. (a) [3] $f(t) = -u(t)t$, $f'(t) = -u(t)$.

(b) [3] $f(t) = u(t)(1-t)$, $f'(t) = -u(t) + \delta(t)$.

(c) [3] $f(t) = (u(t) - u(t-1))(2t-1)$, $f'(t) = 2(u(t) - u(t-1)) - \delta(t) - \delta(t-1)$.

(d) [3] $f(t) = (u(t)-u(t-1))t + (u(t-1)-u(t-2))(t-1) + (u(t-1)-u(t-2))(t-2) + \cdots = u(t)t - u(t-1) - u(t-2) - \cdots$.

24. (a) [4] The roots of the characteristic polynomial are $-1 \pm i$, so the general solution to the homogeneous equation is $e^{-t}(a\cos t + b\sin t)$. The unit impulse response for this second order operator has $w(0) = 0$ and $\dot{w}(0+) = \frac{1}{2}$. The first forces $a = 0$ and the second gives $b = \frac{1}{2}$: $w(t) = \frac{1}{2}u(t)e^{-t}\sin t$.

(b) [4] For $t > 0$, the unit step response is a solution to $p(D)x = 1$. In our case, $x_p = \frac{1}{4}$ is such a solution, and the general solution is then $x = \frac{1}{4} + e^{-t}(a\cos t + b\sin t)$. We require rest initial conditions: $0 = x(0) = \frac{1}{4} + a$ or $a = -\frac{1}{4}$. $\dot{x} = e^{-t}((-a+b)\cos t + (-a-b)\sin t)$, so $0 = \dot{x}(0) = -a + b$ and $b = -\frac{1}{4}$ as well: $v = \frac{1}{4}u(t)(1 - e^{-t}(\cos t + \sin t))$.

(c) [4] $\dot{v} = -\frac{1}{4}e^{-t}((-1+1)\cos t + (-1-1)\sin t) = \frac{1}{2}e^{-t}\sin t$.

(d) (i) [4] This function has a jump in *value*, so the operator must be of first order. $(aD + bI)(2u) = 2a\delta(t) + 2bu(t)$, so $b = 0$ and $a = \frac{1}{2}$: $p(D) = \frac{1}{2}D$.

(ii) [4] This function has no jump but its derivative does, so the operator must be of second order. For $t > 0$, $w(t) = t$ is the solution to $a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$ with $x(0) = 0$ and $\dot{x}(0) = \frac{1}{a_2}$. Plug in: $a_1 + a_0 t = 0$ implies $a_1 = a_0 = 0$, and $1 = \frac{d}{dt}t|_{t=0} = \frac{1}{a_2}$ implies that $a_2 = 1$. So $p(D) = D^2$. Or you can argue that $w(t) = u(t)t$, $\dot{w}(t) = u(t)$ and $\ddot{w}(t) = \delta(t)$, so $a_2\delta(t) = \delta(t)$ and $a_2 = 1$.

(iii) [4] This function $w(t)$ has no jump in value or derivative, but its second derivative does jump: $\ddot{w}(t) = 2u(t)$. So $w^{(3)}(t) = 2\delta(t)$. This means that we are looking for a third order operator, $a_3D^3 + a_2D^2 + a_1D + a_0I$. t^2 is a solution to the homogeneous equation, so $a_2 \cdot 2 + a_1 \cdot 2t + a_0t^2 = 0$, which implies that $a_0 = a_1 = a_2 = 0$. $\ddot{w}(0) = 2$ implies that $a_3 = \frac{1}{2}$ and $p(D) = \frac{1}{2}D^3$. Or you can argue that $w(t) = u(t)t^2$, $\dot{w}(t) = u(t)2t$, $\ddot{w}(t) = 2u(t)$, $w^{(3)}(t) = 2\delta(t)$, so $a_3w^{(3)}(t) = \delta(t)$ implies that $a_3 = \frac{1}{2}$.

25. (a) [6] $x(t) = w(t)*q(t) = \int_0^t w(t-\tau)q(\tau)\,d\tau = \int_0^t e^{-k(t-\tau)}\cos(\omega\tau)\,d\tau = e^{-kt}\int_0^t \mathrm{Re}(e^{(k+i\omega)\tau})\,d\tau = e^{-kt}\mathrm{Re}\frac{e^{(k+i\omega)t}-1}{k+i\omega} = \frac{1}{k^2+\omega^2}\mathrm{Re}((k-i\omega)((\cos(\omega t) - e^{-kt}) + i\sin(\omega t)) = \frac{1}{k^2+\omega^2}(k\cos(\omega t) + \omega\sin(\omega t) - ke^{-kt})$. Then $\dot{x} = \frac{1}{k^2+\omega^2}(-k\omega\sin(\omega t) + \omega^2\cos(\omega t) + k^2e^{-kt})$, and indeed $\dot{x} + kx = \cos(\omega t)$. Also, $x(0) = 0$: the convolution chose the transient just right.

(b) [6] $x(t) = w(t) * q(t) = \int_0^t w(t-\tau)q(\tau)\,d\tau = \frac{1}{\omega_n}\int_0^t \sin(\omega_n(t-\tau))\,d\tau = \frac{1}{\omega_n^2}\cos(\omega_n(t-\tau))|_0^t = \frac{1}{\omega_n^2}(1-\cos(\omega_n t))$. Then $\dot{x} = \frac{1}{\omega_n}\sin(\omega_n t)$ and $\ddot{x} = \cos(\omega_n t)$, so it is true that $\ddot{x} + \omega_n^2 x = 1$. Also $x(0) = 0$ and $\dot{x}(0) = 0$: so rest initial conditions. Once again the convolution integral has chosen just the right homogeneous solution to produce rest initial conditions.

(c) [6] $t^2 * t = \int_0^t (t-\tau)^2\tau\,d\tau = \int_0^t (t^2\tau - 2t\tau^2 + \tau^3)\,d\tau = \frac{1}{2}t^4 - \frac{2}{3}t^4 + \frac{1}{4}t^4 = \frac{1}{12}t^4$. $t * t^2 = \int_0^t (t-\tau)\tau^2\,d\tau = \int_0^t (t\tau^2 - \tau^3)\,d\tau = \frac{1}{3}t^4 - \frac{1}{4}t^4 = \frac{1}{12}t^4$.

(d) [6] $t * t = \int_0^t (t-\tau)\tau\,d\tau = \int_0^t (t\tau - \tau^2)\,d\tau = \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{1}{6}t^3$.

Now $(t*t)*t = \frac{1}{6}\int_0^t (t-\tau)^3\tau\,d\tau = \frac{1}{6}\int_0^t (t^3 - 3t^2\tau + 3t\tau^2 - \tau^3)\tau\,d\tau = \frac{1}{6}(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5})t^5 = \frac{1}{120}t^5$, while $t * (t * t) = \frac{1}{6}\int_0^t (t-\tau)\tau^3\,d\tau = \frac{1}{6}(\frac{1}{4} - \frac{1}{5})t^5 = \frac{1}{120}t^5$.