CHRISTINE BREINER: Welcome back to recitation. In this video I would like us to use Green's theorem to compute the following integral, where it's the integral over the curve C, where C is the circle drawn here. So the circle is oriented so the interior is on the left and it's centered at the point where x equals a, y equals 0. So the integral-- you can read it but I will read it also for you-- it's the integral of 3 x squared y squared dx plus 2 x squared times the quantity 1 plus x*y dy. So you are supposed to use Green's theorem to compute that. And why don't you pause the video, give it a shot, and then when you're ready to see my solution you can bring the video back up.

OK, welcome back. Well, what we're going to do, obviously to use Green's theorem to compute this-- because I gave you a big hint that you're supposed to use Green's theorem, because we have an integral over a closed curve. And what we're going to do is now take the integral of another certain integrand-- obviously related in some way to this-- over the region that I will shade here. So we're interested in this shaded region now.

So let me write down what Green's theorem is and then we'll put in the important parts. OK, so just to remind you, I'm not going to write down all the hypotheses of Green's theorem that we need, but the point that I want to make is that if we start the integral over this closed curve C of M*dx plus N*dy we can also integrate over the region that C encloses of N sub x minus M sub y dy dx. So that is ultimately what we're going to do.

And so in this case again, as always, M is going to be the function associated with the dx portion. It's the i-th component of the vector field. And N is going to be the function associated here with the dy. That's standard obviously. So now what we want to do is transform what we have there into something that looks like this. So the region-- I'm just going to keep calling it R for the moment-- but the region R, you'll notice, is the thing that I shaded in the drawing.

And so now let's compute what this is-- well you have the capacity to do that. That's just some straight taking derivative. So I'm just going to write down what it is and not show you all the individual pieces. So I'll just write down what you get and then the simplified version. So you get 6 x squared y plus 4x is N sub x. And then M sub y is negative 6 x squared y. And I'm going to call dy*dx, I'm just going to refer to it now as dA, because that makes it a lot easier to write. Oh, I guess I should call this-- well, dA, this is the area. The volume form there. So this simplifies and I just get the integral of 4x over this region dA.

Now this-- you saw an example of this in lecture of how to deal with these types of problems.

At this point, if I really wanted to, I could figure out what the bounds are in that region R in terms of x and y, and I could do a lot of work and integrate it all, or I could remember one simple fact. Which is that if I have-- I think you see it in class as x bar-- the center of mass should be equal to 1 over the volume of the region times the integral of x dA over the region. That's what we know. This should be-- this maybe doesn't look like a V. But so in this case, volume is area, isn't it? Maybe I should write area, that might make you nervous.

So if I take the area-- I could just say A of R-- if I take the area, 1 over the area, and then I multiply by the integral of x over R with dy*dx, with respect to dA, then I get the center of mass. Well, let's look at what-- in this picture, what is the center of mass here? If I want to balance this thing on a pencil tip, if I want to balance this disc-- assuming the density is everywhere the same-- on a pencil tip, where am I going to put the pencil? I'm going to put it right at the center. The x-value there is a, the y-value there is 0. So if I had the y center of mass, I would want 0. But I want the x center of mass, so I want a.

This x bar is actually equal to, from the picture, is equal to a. And now let's notice what I've done. I have taken-- I had this quantity here-- I've taken this quantity except for the 4 and I have a way of writing explicitly what this quantity is without actually doing any of the integration. I haven't done any sophisticated things at this point except know what the center of mass is.

So also what is the area of the region? I'm going to need that. The area of the region-- it's a circle of radius a. So the area is pi a squared. So this quantity is pi a squared. And all this together tells me that the integral of x over R dA, if I solve for this part, I get a times pi a squared. So I get pi a cubed. And that only differs from our answer by one thing. There's a scalar-- you multiply by 4 and that gives us what we want. I said differs from our answer. Differs from what we want by one thing. We just multiplied by 4 there, so we multiply by 4 here.

So in fact-- maybe there looks like there was a little magic here, so let me point out some of the key points at the end. I started off knowing I was trying to integrate 4x over the region R. And R in this case was a circle centered at (a, 0) and of radius a. And then I said, well, I don't want to do a lot of work for this. So I'm going to not cheat, but I'm going to use my knowledge of the center of mass to make this easier.

So the center of mass is equal to 1 divided by the area of the region times the integral of x over the region. So I want to find the integral of x over the region, I just solve for the integral of x over the region. I just solve for that part. The center of mass-- from just looking at the picture and understanding what the center of mass means-- the center of mass in the x component is a. The area is pi a squared. So I end up with a pi a cubed when I solve for the integral of x. And then because I wanted the integral of 4x, I just multiply by 4. And so the final answer is actually 4 pi a cubed.

So what I was trying to find, if you remember, was I was trying to find the value, when I integrated over this curve, of a certain vector field. And that one was going to be a little messy to do it that way. But Green's theorem, actually there's a lot of cancellation which makes it much easier, and then the center of mass is a nice little trick to use at the end. And then the calculation is quite simple. So I think that's where I'll stop.