Work integrals

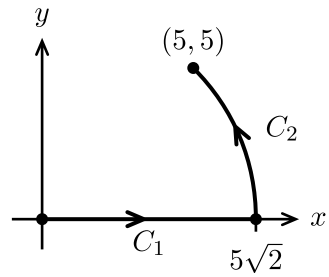
1. Let C be the path from (0,0) to (5,5) consisting of the straight line from (0,0) to $(5\sqrt{2}, 0)$ followed by the arc from $(5\sqrt{2}, 0)$ to (5,5) that is part of the circle of radius $5\sqrt{2}$ centered at the origin.

Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following vector fields $\mathbf{F}$

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; b) $\mathbf{F} = x\,\mathbf{j}$.

(Remember to work smart and exploit geometry where possible.)

<u>**Answer:**</u> We have $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.



a) We note that $\mathbf{F} \perp C_2$ everywhere, so $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \;\Rightarrow\; \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

On C_1 we have $y = 0$, so $dy = 0$, and x runs from 0 to $5\sqrt{2}$. Taking $M = x$, $N = y$ we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} M\,dx + N\,dy = \int_{C_1} x\,dx + y\,dy = \int_0^{5\sqrt{2}} x\,dx = 25.$$

The work integral $= 25$.

b) In this case, $\mathbf{F} \perp C_1$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} x\,dy.$$

We parametrize C_2 by $x = 5\sqrt{2}\cos t$; $y = 5\sqrt{2}\sin t$; $0 \le t \le \pi/4$. This gives

$$\int_{C_2} x\,dy = \int_0^{\pi/4} 50\cos^2 t\,dt = \int_0^{\pi/4} 50\left(\frac{1+\cos 2t}{2}\right)dt = \frac{25\pi}{4} + \frac{25}{2}.$$

The work integral is $\dfrac{25\pi}{4} + \dfrac{25}{2}$.

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