CHRISTINE BREINER: Welcome back to recitation. In this video I want to show how we can use change of variables, to polar coordinates in particular, to evaluate an integral that, without the change in variables, we don't have the techniques to do. So I'm going to show us how to evaluate the integral from minus infinity to infinity of e to the minus x squared dx. And I'll just point out that if you do anything in probability you will see this integral a great deal. So this is a distribution and you'll see it a great deal if you ever do anything in probability theory.

But how are we going to use polar coordinates to evaluate this? Well, the object is going to be to introduce a little bit more into this integral, so that when I actually introduce that in I'm going to have an r squared term in the exponent and I'm going to have-- of course by the change of variables in the Jacobian-- multiplied by r. And that's what's going to save us.

So let me actually just get right to it. I'm going to call this integral, this quantity, capital I. We'll just call this quantity capital I. And then what I'm going to do is introduce a couple more things. So let me write it down, and then I will justify that it's reasonable to even look at this thing. And that it will give us something useful.

OK, so I just took this integral and what I've done is I've now taken this quantity and I've multiplied by an e to the minus y squared. And then I'm now integrating from minus infinity to infinity in dy also. And what I want to point out, that maybe isn't immediately obvious from the way it's written here, but will be immediately obvious when I write it a different way, is that this quantity is actually just the square of this quantity. So it's actually just two of these multiplied together.

And let me point out why that is. Because this term right here can be rewritten as e to the minus x squared, e to the minus y squared. And then I have two integrals, which are from minus infinity to infinity both. Sorry, that one looks a little off kilter. But those two integrals, which the bounds are minus infinity to infinity for both. And then I have a dx and a dy.

Now the reason that it is just I squared-- this quantity is just I squared-- is because if you notice, this first integral, the inside integral, this is independent of x. So I can move it out. So let me actually come up here and write that down. So I can move it out and I can actually rewrite the integral. And I'll put the bounds here, because it's high up enough that I can actually write it. e to the minus y squared. And then integral from minus infinity to infinity e to the minus x squared dx dy. Right?

So let's point out again what I did. This quantity I circled was the constant when I considered x the variable. Right? In terms of x, this is the constant, so I can move it in front of that integral, which is the dx integral. And so now what do I see? Well, this quantity is what I've called I. And so now that's a constant. So I can move that out to the front. That's a constant. That's I, as designated.

And so now I have the integral for minus infinity to infinity of e to the minus y squared dy. And then that again is another I. Because if you notice, I mean, this was e to the minus x squared dx, and so if I keep this as e to the minus y squared dy, it's going to be exactly the same value. So this is actually equal to I squared.

So the point I want to make, if you come back over here to this integral, is that this quantity I'm going to be able to integrate very easily. And when I do that, if I take the square root, I'm going to get this value. Because I just showed this thing in the box was equal to I squared. So let me write it down. On this last part of the board I will write down that thing again and then we'll actually evaluate it and we'll see what we get.

So let me rewrite. This is the thing we want to evaluate. And I'm going to remind myself that that's equal to I squared. I squared is equal to this quantity. And I mentioned at the beginning that we're going to use a trick. We're going to change variables into polar coordinates. And so notice that the region we're integrating over is the entire xy-plane. And in polar coordinates, what's that going to be? Theta is going to run from 0 all the way around to 2 pi, and r is going to run from 0 to infinity. And so those will be our bounds for r and theta. Because if you want to get the entire xy-plane, in terms of r and theta, that is what you have to do.

And I want to mention also, what is this quantity going to become in terms of r and theta? Well, notice that this is e to the minus x squared minus y squared. It's just really e to the minus quantity x squared plus y squared. It's really e to the minus r squared. So what we get when we do a change of variables, is we do-- I'll put the theta on the inside actually, because that'll be easy to do first. So the r is going to be on the outside. The theta is going to be on the inside. And then e to the minus r squared, that's a direct substitution. x squared plus y squared equals r squared. And then I get r d theta dr.

And this comes from the Jacobian that you computed-- I believe in lecture even-- to show how you change from x, y variables to r, theta variables. And I put the d theta first here because I wanted to integrate in theta first. And the reason I want to integrate in theta first is notice that--

because nothing here depends on theta-- all I pick up is a 2 pi. I just pick up-- when I integrate, I get a theta. I evaluate it at 2 pi, and then I evaluate it at 0 and I take the difference. And so if I do that line all I get is-- well, I should probably write it in front-- all I get is 2 pi from the theta, and then the integral from 0 to infinity, e to the minus r squared times r dr.

Now this is a much easier quantity to evaluate than e to the minus x squared dx. Because now we have an r here. So now it's a natural substitution type of problem. And we can do it right away. I'm going to write it down and then we'll check and make sure I didn't make a mistake. We should get something like e to the minus r squared over 2 with a negative sign in front.

So let me make sure when I take this derivative-- you could just do a substitution to check, but you should get exactly this kind of thing. When I take the derivative of e to the minus r squared, I get a negative 2r and then-- did I do something wrong here? Oh yeah. I'm good. I get a negative 2r. And so the negatives kill off, the 2s divide out, and I get my e to the minus r squared times r. So that's good. And now I have to evaluate it at the bounds.

So again, this was much easier to evaluate if I do a substitution and I let r-- I think I want to let r squared equal u or something like this. I don't even know. I just did the substitution without thinking about it. So you can figure out what you need to substitute. But this is what you get. And so let me actually just evaluate at the bounds and see what happens.

So as r goes to infinity, I get e to the minus r squared. As r goes to infinity, e to the minus r squared goes to 0. And so the first term is 0 when I evaluate. And then so the second term I get is, I do 0 minus 2 pi times-- if I evaluate at e to the 0 I get 1-- so I get negative 1 over 2. The negative 1 comes from e to the 0. And then I divide by 2 so I get my 2 there. So a negative and a negative gives me a positive. 2 divided by 2. And so the whole thing equals pi. Whew! Just enough room.

And so I just want to remind us where we came from. We came from pi-- we wanted to show was equal-- well, the pi, where did it come from? It's equal, if we go all the way back up, to I squared. And so we wanted to show what I was equal to. I then is equal to just the square root of pi. So I can come back over to where I started, which is over here. And I can say this is equal to the square root of pi.

And again, I just want to mention what it came down to. It came down to introducing a little bit more. Essentially we took something that was a single variable problem, we made it a

multivariable problem. But what that allowed us to do is to do a change of coordinates from x, y into the polar coordinate system. And the trick here-- as you come through-- the trick where it actually happens is right here. Is because this is easy to integrate now that I have an r here. If I didn't have-- my problem initially was I didn't have an x when I was trying to find an antiderivative. But when I do the change of variables, I get an e to the minus r squared. And then with an r here, that's much easier to find an antiderivative. OK. So I think that is where I'll stop.