18.02 Problem Set 5, Part II Solutions

Problem 1 $R = f(r, w) = kwr^{-4}$ (k constant)

(a) $dR = f_r dr + f_w dw = k(w(-4r^{-5})dr + r^{-4}dw)$.

(b) $\frac{dR}{R} = -4\frac{dr}{r} + \frac{dw}{w}$.

(c) $\frac{dR}{R}$ is more sensitive to $\frac{dr}{r}$ = relative change in r. Opposite signs in $\frac{dr}{r}$, $\frac{dw}{w}$ (or in dr and dw, since $r, w > 0$) will cause errors to add.

Problem 2 $\frac{Df}{Dt} = \frac{d}{dt} f(\mathbf{r}(t), t) = \frac{d}{dt} f(x(t), y(t), z(t), t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} = \mathbf{r}'(t) \cdot \nabla f(\mathbf{r}(t)) + \frac{\partial f}{\partial t} = \mathbf{v} \cdot \nabla f + \frac{\partial f}{\partial t}$ using $\mathbf{v} = \mathbf{r}'(t)$

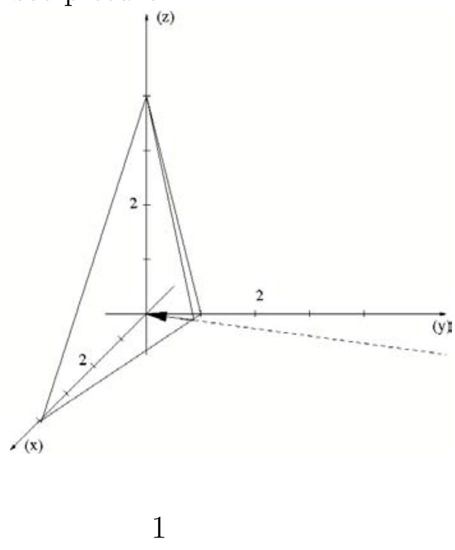
Problem 3 $\dfrac{D\rho}{Dt} = \dfrac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$

(a) If $\rho = \rho(t)$ only, then $\nabla\rho = \langle \rho_x, \rho_y, \rho_z \rangle = \mathbf{0}$. Thus $\frac{D\rho}{Dt} = 0$ if and only if $\frac{\partial\rho}{\partial t} = 0$.

b) If $\frac{\partial\rho}{\partial t} = 0$, then $\frac{D\rho}{Dt} = 0$ if and only if $\mathbf{v} \cdot \nabla\rho = 0$. So the condition for stratified flow is that the velocity vectors of the flow are orthogonal to the density gradients, or, equivalently, tangent to the surfaces of constant density.

c) If $\rho = \rho(y)$ only, then $\nabla\rho = \langle 0, \rho_y \rangle$, so that the gradient of the density is always parallel to $\mathbf{j}$. Therefore, by the result of part(b), the streamlines, which follow the velocity vectors $\mathbf{v}$, are always *horizontal*. The flow is thus layered by density, which is consistent with the meaning of the word stratified.

Problem 4. (a) and (e) – see picture:



(b) We compute

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle -1,\ -4 \rangle\ .$$

(c) The level curve for $f = 0$ is given by

$$x + 4y = 4\ .$$

We are looking for a point (x, y) that lies on the line that passes through the origin in gradient direction, i.e.,

$$\langle x, y \rangle = \langle 0, 0 \rangle + s\ \langle -1, -4 \rangle\ .$$

Thus $x = -s$ and $y = -4s = 4x$. Plugging $y = 4x$ into the level curve for $f = 0$ gives

$$x + 16x = 4\ ,$$

or $x = 4/17$ and $y = 16/17$.

(d) The directional derivative is given by

$$\nabla f(x,y) \cdot \frac{\mathbf{w}}{|\mathbf{w}|} = \langle -1,\ -4 \rangle \cdot \frac{\langle -2, -1 \rangle}{\sqrt{5}} = \frac{6}{\sqrt{5}}\ .$$