18.02 Problem Set 5

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (15 points)

At MIT the underlined problems must be done and turned in for grading.
The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : $\underline{2}$, 5$\underline{\text{b}}$, $\underline{10}$' means do the indicated problems from supplementary problems section 1B.

1 Differentials. Chain rule
§2C: 1$\underline{\text{ad}}$, $\underline{2}$, $\underline{3}$, 5$\underline{\text{ab}}$ Others: 1bc
§2E: 1$\underline{\text{c}}$, 2$\underline{\text{bc}}$, 8$\underline{\text{a}}$; Others: 1ab, 2d, 4, 5, 7

2 Gradient and directional derivatives
§2D: 1$\underline{\text{ae}}$, 2$\underline{\text{b}}$, 3$\underline{\text{a}}$, $\underline{8}$, $\underline{9}$; Others: 1bc, 2a, 3b, 4, 5
§2E: $\underline{7}$

Part II (17 points)

Problem 1 (4: 1,2,1)
In laminar flow in a cylinder (for example, blood flow in a vein or artery), the resistance R to the flow is related to the length w and radius r of the cylinder by the law of Poiseuille: $R = k\dfrac{w}{r^4}$ for some constant k.

a) Compute the linear approximation $\mathrm{d}R$ to the change in R, in terms of the changes in w and r.

b) Compute the linear approximation $\dfrac{\mathrm{d}R}{R}$ to the *relative* change in R in terms of $\dfrac{\mathrm{d}w}{w}$ = the relative change in w and $\dfrac{\mathrm{d}r}{r}$ = the relative change in r.

c) For relative changes in w and r of about the same sizes, which variable contributes more to the relative change in R ? Also, in order to produce the greatest relative change in R, should the changes in w and r both have the same sign or opposite signs (and why)?

Problem 2 (3)
Let $f(x, y, z, t)$ be a smooth function, and let $\nabla f = \langle f_x, f_y, f_z\rangle$ be the gradient in the *space* variables only. Let $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a smooth curve, and $\mathbf{v} = \mathbf{r}'(t)$; and suppose we use the notation $\dfrac{Df}{Dt} = \dfrac{d}{dt} f(\mathbf{r}(t), t)$.

Use the Chain Rule to show that $\dfrac{Df}{Dt} = \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$.

Background: The notation $\frac{D}{Dt}$ comes from the physics of fluid motion, where it is called the *convective derivative* (or material or substantial derivative, and by several other names), and means the rate of change along a moving path of some physical quantity (scalar or vector) which is being transported by fluid currents.

In this macroscopic model, the fluid is pictured as a continuum of point masses rather than as individual molecules. At a location (x, y, z) in space and a time t, the point mass has a density $\rho = \rho(x, y, z, t)$, and a velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$. This means that the vector $\mathbf{v}(x, y, z, t)$ points in direction tangent to the path of a particle at (x, y, z, t) in the flow, and has magnitude equal to the instantaneous speed of the particle located at that point and which is moving in the flow.

Now suppose that the curve $\mathbf{r} = \mathbf{r}(t)$ is a path of a point mass in the flow, so that (by definition) $\mathbf{r}'(t) = \mathbf{v}(\mathbf{r}(t), t)$. The convective derivative $\frac{Df}{Dt}$ of f along this path is the time rate of change of f using *only* the values of $f(x, y, z, t)$ for which the space variables (x, y, z) are *restricted to the path* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ of a particle in the flow. For this reason you will see the convective derivative described as the rate of change of the quantity f “moving along the flow” or “moving with an element of the fluid” (and other similar language).

Problem 3 (5: 1,2,2) (*continuation*)
Now take the case $f = \rho$, the density of the fluid. A fluid flow is called *incompressible* if $\dfrac{D\rho}{Dt} = 0.$

As discussed above, this means that the mass density is constant along the paths of the flow. Any substance (like water, at moderate pressures) which has the property that its density is constant in all variables (x, y, z, t) will of course be incompressible, which is the usual way one pictures something which cannot be compressed. However, incompressibility is in general a property of the *flow* rather than just the fluid itself, since it says only that the rate of change of the density moving along the flow is zero. The following examples illustrate this.

a) Suppose that the density function depends only on *time* t but is constant in the space variables (x, y, z), that is, $\rho = \rho(t)$. Then show that the flow is incompressible if and only if the density $\rho(t)$ is constant in all the variables (x, y, z, t) (that is, the constant-density case discussed above).

b) Next suppose instead that the density depends only on the *space* variables (x, y, z) but not (explicity) on t, so that $\rho = \rho(x, y, z)$. An incompressible flow in this case is called *stratified*.
Use the result of problem 2 to give the condition on ρ and $\mathbf{v}$ for stratified flow.

A flow is called *steady* if the density ρ and the velocity field $\mathbf{v}$ of the flow do not depend explicitly on the time t, i.e. $\rho = \rho(x, y, z)$ and $\mathbf{v} = \mathbf{v}(x, y, z)$. In this case, the term *streamlines* is used for the paths of the particles in the flow, since they keep their same shapes over time.

c) Suppose one has a 2D stratified steady flow, so that $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$, and suppose also that the density varies only with the height y. Draw a picture of the streamlines for such as flow. Then explain why they must follow this pattern, and why the term "stratified" fits in this case.
(This could be, for example, a cross-section of a very regular ocean current, if it is an incompressible steady flow whose density varies only with the depth.)

Problem 4 (5: 1,1,1,1,1)
For the linear function $f(x, y) = 4 - x - 4y$:
a) Sketch the portion of the graph in the first octant
b) Compute the gradient of f.
c) Find the point on the level curve $f(x, y) = 0$ such that the line in the gradient direction passes through the origin, and then sketch in the gradient at that point.
d) Compute the directional derivative of f in the direction $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$
e) Sketch in the slope triangles for the rates of change of f in the gradient direction and in the direction of $\mathbf{w}$.