Distances to planes and lines

In this note we will look at distances to planes and lines. Our approach is geometric. Very broadly, we will draw a sketch and use vector techniques.

Please note is that our sketches are not oriented, drawn to scale or drawn in perspective. Rather they are a simple 'cartoon' which shows the important features of the problem.

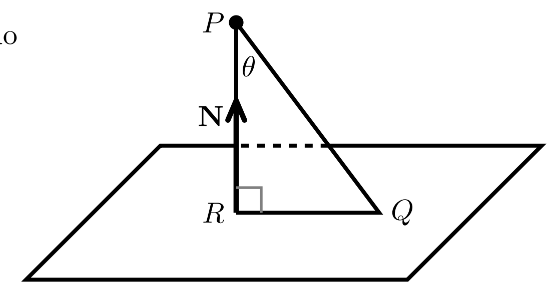
1. *Distance: point to plane:*
Ingredients: i) A point P, ii) A plane with normal $\overrightarrow{\mathbf{N}}$ and containing a point Q.

The distance from P to the plane is $d = |\overrightarrow{\mathbf{PQ}}| \cos\theta = \left| \overrightarrow{\mathbf{PQ}} \cdot \frac{\overrightarrow{\mathbf{N}}}{|\mathbf{N}|} \right|.$

We will explain this formula by way of the following example.

Example 1: Let $P = (1, 3, 2)$. Find the distance from P to the plane $x + 2y = 3$.

<u>**Answer:**</u> First we gather our ingredients.
$Q = (3, 0, 0)$ is a point on the plane (it is easy to find such a point).
$\mathbf{N}$ = normal to plane = $\mathbf{i} + 2\mathbf{j}$.
R = point on plane closest to P (this is point unknown and we do not need to find it to find the distance). The figure shows that



$$\text{distance} = |PR| = \left|\overrightarrow{\mathbf{PQ}}\right| \cos\theta = \left| \overrightarrow{\mathbf{PQ}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|} \right|.$$

Computing $\overrightarrow{\mathbf{PQ}} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ gives

$$\text{distance} = \left| \overrightarrow{\mathbf{PQ}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|} \right| = \left| \langle 2, -3, -2 \rangle \cdot \frac{\langle 1, 2, 0 \rangle}{\sqrt{5}} \right| = \frac{4}{\sqrt{5}}.$$

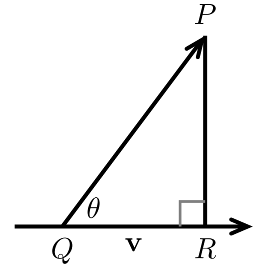
2. *Distance: point to line:*
Ingredients: i) A point P, ii) A line with direction vector $\mathbf{v}$ and containing a point Q.

The distance from P to the line is $d = |\mathbf{QP}| \sin\theta = \left| \overrightarrow{\mathbf{QP}} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right|.$

We will explain this formula by way of the following example.

Example 2: Let $P = (1, 3, 2)$, find the distance from the point P to the line through $(1, 0, 0)$ and $(1, 2, 0)$.

<u>**Answer:**</u> First we gather our ingredients.
$Q = (1, 0, 0)$ (this is easy to find).
$\mathbf{v} = \langle 1,\ 2,\ 0 \rangle - \langle 1,\ 0,\ 0 \rangle = 2\mathbf{j}$ is parallel to the line.
R = point on line closest to P (this is point is unknown).
Using the relation $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$, the figure shows that
$\text{distance} = |PR| = \left|\overrightarrow{\mathbf{PQ}}\right| \sin\theta = \left| \overrightarrow{\mathbf{QP}} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right|.$



Computing: $\overrightarrow{\mathbf{PQ}} = 3\mathbf{j} + 2\mathbf{k}$, which implies $\left| \overrightarrow{\mathbf{QP}} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right| = |(3\mathbf{j} + 2\mathbf{k}) \times \mathbf{j}| = |-2\mathbf{i}| = 2.$

3. *Distance between parallel planes:*

The trick here is to reduce it to the distance from a point to a plane.

Example 3: Find the distance between the planes $x + 2y - z = 4$ and $x + 2y - z = 3$.

Both planes have normal $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ so they are parallel.

Take any point on the first plane, say, $P = (4, 0, 0)$.

Distance between planes = distance from P to second plane.

Choose $Q = (1, 0, 0)$ = point on second plane

$\Rightarrow\ d = |\overrightarrow{\mathbf{QP}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}| = |3\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})|/\sqrt{6} = \sqrt{6}/2.$

4. *Distance between skew lines:*

We place the lines in parallel planes and find the distance between the planes as in the previous example

As usual it's easy to find a point on each line. Thus, to find the parallel planes we only need to find the normal.

$$\mathbf{N} = \mathbf{v_1} \times \mathbf{v_2},$$

where $\mathbf{v_1}$ and $\mathbf{v_2}$ are the direction vectors of the lines.