1. Vectors and Matrices

1A. Vectors

Definition. *A* **direction** *is just a unit vector. The* **direction of A** *is defined by*

$$\text{dir}\ \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \qquad (\mathbf{A} \neq \mathbf{0});$$

it is the unit vector lying along $\mathbf{A}$ *and pointed like* $\mathbf{A}$ *(not like* $-\mathbf{A}$*).*

1A-1 Find the magnitude and direction (see the definition above) of the vectors

a) $\mathbf{i}+\mathbf{j}+\mathbf{k}$ b) $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ c) $3\mathbf{i}-6\mathbf{j}-2\mathbf{k}$

1A-2 For what value(s) of c will $\frac{1}{5}\mathbf{i}-\frac{1}{5}\mathbf{j}+c\,\mathbf{k}$ be a unit vector?

1A-3 a) If $P=(1,3,-1)$ and $Q=(0,1,1)$, find $\mathbf{A}=PQ$, $|\mathbf{A}|$, and dir $\mathbf{A}$.

b) A vector $\mathbf{A}$ has magnitude 6 and direction $(\mathbf{i}+2\mathbf{j}-2\mathbf{k})/3$. If its tail is at $(-2,0,1)$, where is its head?

1A-4 a) Let P and Q be two points in space, and X the midpoint of the line segment PQ. Let O be an arbitrary fixed point; show that as vectors, $OX=\frac{1}{2}(OP+OQ)$.

b) With the notation of part (a), assume that X divides the line segment PQ in the ratio $r:s$, where $r+s=1$. Derive an expression for OX in terms of OP and OQ.

1A-5 What are the $\mathbf{i}\,\mathbf{j}$-components of a plane vector $\mathbf{A}$ of length 3, if it makes an angle of 30^o with $\mathbf{i}$ and 60^o with $\mathbf{j}$. Is the second condition redundant?

1A-6 A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the $\mathbf{i}\,\mathbf{j}$-components).

1A-7 Let $\mathbf{A}=a\,\mathbf{i}+b\,\mathbf{j}$ be a plane vector; find in terms of a and b the vectors $\mathbf{A}'$ and $\mathbf{A}''$ resulting from rotating $\mathbf{A}$ by 90^o a) clockwise b) counterclockwise.

(Hint: make $\mathbf{A}$ the diagonal of a rectangle with sides on the x and y-axes, and rotate the whole rectangle.)

c) Let $\mathbf{i}'=(3\,\mathbf{i}+4\,\mathbf{j})/5$. Show that $\mathbf{i}'$ is a unit vector, and use the first part of the exercise to find a vector $\mathbf{j}'$ such that $\mathbf{i}', \mathbf{j}'$ forms a right-handed coordinate system.

1A-8 The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let $\mathbf{A}=a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$ be a space vector, represented as an origin vector, and let α, β, and γ be the three angles ($\leq \pi$) that $\mathbf{A}$ makes respectively with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

a) Show that dir $\mathbf{A} = \cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k}$. (The three coefficients are called the *direction cosines* of $\mathbf{A}$.)

b) Express the direction cosines of $\mathbf{A}$ in terms of a, b, c; find the direction cosines of the vector $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

c) Prove that three numbers t, u, v are the direction cosines of a vector in space if and only if they satisfy $t^2+u^2+v^2=1$.

1A-9 Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides **A** and **B**.)

1A-10 Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

1A-11 Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show $X = Y$.)

1A-12* Label the four vertices of a parallelogram in counterclockwise order as OPQR. Prove that the line segment from O to the midpoint of PQ intersects the diagonal PR in a point X that is 1/3 of the way from P to R.

(Let $\mathbf{A} =$ OP, and $\mathbf{B} =$ OR; express everything in terms of **A** and **B**.)

1A-13* a) Take a triangle PQR in the plane; prove that as vectors $PQ + QR + RP = \mathbf{0}$.

b) Continuing part a), let **A** be a vector the same length as PQ, but perpendicular to it, and pointing outside the triangle. Using similar vectors **B** and **C** for the other two sides, prove that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. (This only takes one sentence, and no computation.)

1A-14* Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn't cross itself (i.e., one whose interior is a single region); label its vertices $P_1, P_2, \ldots, P_n$ as you walk around it.

1A-15* Let $P_1, \ldots, P_n$ be the vertices of a regular n-gon in the plane, and O its center; show without computation or coordinates that $OP_1 + OP_2 + \ldots + OP_n = \mathbf{0}$,

a) if n is even; b) if n is odd.

1B. Dot Product

1B-1 Find the angle between the vectors

a) $\mathbf{i} - \mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ b) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

1B-2 Tell for what values of c the vectors $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ will

a) be orthogonal b) form an acute angle

1B-3 Using vectors, find the angle between a longest diagonal PQ of a cube, and

a) a diagonal PR of one of its faces; b) an edge PS of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

1B-4 Three points in space are $P : (a, 1, -1)$, $Q : (0, 1, 1)$, $R : (a, -1, 3)$. For what value(s) of a will PQR be

a) a right angle b) an acute angle ?

1B-5 Find the component of the force $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in

a) the direction $\dfrac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}$ b) the direction of the vector $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

1B-6 Let O be the origin, c a given number, and $\mathbf{u}$ a given direction (i.e., a unit vector). Describe geometrically the locus of all points P in space that satisfy the vector equation

$$OP \cdot \mathbf{u} \; = \; c|OP| \; .$$

In particular, tell for what value(s) of c the locus will be (Hint: divide through by $|OP|$):

a) a plane b) a ray (i.e., a half-line) c) empty

1B-7 a) Verify that $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ and $\mathbf{j}' = \dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ are perpendicular unit vectors that form a right-handed coordinate system

b) Express the vector $\mathbf{A} = 2\,\mathbf{i} - 3\,\mathbf{j}$ in the $\mathbf{i}'\mathbf{j}'$-system by using the dot product.

c) Do b) a different way, by solving for $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{i}'$ and $\mathbf{j}'$ and then substituting into the expression for $\mathbf{A}$.

1B-8 The vectors $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$, $\mathbf{j}' = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$, and $\mathbf{k}' = \dfrac{\mathbf{i}+\mathbf{j}-2\,\mathbf{k}}{\sqrt{6}}$ are three mutually perpendicular unit vectors that form a right-handed coordinate system.

a) Verify this. b) Express $\mathbf{A} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ in this system (cf. 1B-7b)

1B-9 Let $\mathbf{A}$ and $\mathbf{B}$ be two plane vectors, neither one of which is a multiple of the other. Express $\mathbf{B}$ as the sum of two vectors, one a multiple of $\mathbf{A}$, and the other perpendicular to $\mathbf{A}$; give the answer in terms of $\mathbf{A}$ and $\mathbf{B}$.

(Hint: let $\mathbf{u} = \text{dir } \mathbf{A}$; what's the $\mathbf{u}$-component of $\mathbf{B}$?)

1B-10 Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

1B-11 Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

1B-12 Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

1B-13 Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

(Hint: consider two unit vectors making angles θ_1 and θ_2 with the positive x-axis.)

1B-14 Prove the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$ by using the algebraic laws for the dot product and its geometric interpretation.

1B-15* The **Cauchy-Schwarz inequality**

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or space; under what circumstances does equality hold?

$$(*) \qquad |\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}| \; .$$

b) If the vectors are plane vectors, write out what this inequality says in terms of $\mathbf{i}\,\mathbf{j}$-components.

c) Give a different argument for the inequality (*) as follows (this argument generalizes to n-dimensional space):

i) for all values of t, we have $(\mathbf{A}+t\mathbf{B})\cdot(\mathbf{A}+t\mathbf{B}) \geq 0$;

ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in t ;

iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

1C. Determinants

1C-1 Calculate the value of the determinants a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

1C-2 Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

a) the first row b) the first column

1C-3 Find the area of the plane triangle whose vertices lie at

a) $(0,0),(1,2),(1,-1)$; b) $(1,2),(1,-1),(2,3)$.

1C-4 Show that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1-x_2)(x_2-x_3)(x_3-x_1)$.

(This type of determinant is called a **Vandermonde** determinant.)

1C-5 a) Show that the value of a 2×2 determinant is unchanged if you add to the second row a scalar multiple of the first row.

b) Same question, with "row" replaced by "column".

1C-6 Use a Laplace expansion and Exercise 5a to show the value of a 3×3 determinant is unchanged if you add to the second row a scalar multiple of the third row.

1C-7 Let (x_1,y_1) and (x_2,y_2) both range over all unit vectors.
Find the maximum value of the function $f(x_1,x_2,y_1,y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

1C-8* The base of a parallelepiped is a parallelogram whose edges are the vectors $\mathbf{b}$ and $\mathbf{c}$, while its third edge is the vector $\mathbf{a}$. (All three vectors have their tail at the same vertex; one calls them "coterminal".)

a) Show that the volume of the parallelepiped $\mathbf{abc}$ is $\pm\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

b) Show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) =$ the determinant whose rows are respectively the components of the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$.

(These two parts prove the volume interpretation of a 3×3 determinant.

1C-9 Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices $(0,0,0), (0,-1,2), (0,1,-1), (1,2,1)$. (The volume of a tetrahedron is $\frac{1}{3}$(base)(height).)

1C-10 Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

1D. Cross Product

1D-1 Find $\mathbf{A} \times \mathbf{B}$ if

a) $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{k},\ \mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

1D-2 Find the area of the triangle in space having its vertices at the points

$$P:(2,0,1),\ Q:(3,1,0),\ R:(-1,1,-1).$$

1D-3 Two vectors $\mathbf{i}'$ and $\mathbf{j}'$ of a right-handed coordinate system are to have the directions respectively of the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find all three vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$.

1D-4 Verify that the cross product $\times$ does not in general satisfy the associative law, by showing that for the particular vectors $\mathbf{i}, \mathbf{i}, \mathbf{j}$, we have $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{j})$.

1D-5 What can you conclude about $\mathbf{A}$ and $\mathbf{B}$

a) if $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$; b) if $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$.

1D-6 Take three faces of a unit cube having a common vertex P; each face has a diagonal ending at P; what is the volume of the parallelepiped having these three diagonals as coterminous edges?

1D-7 Find the volume of the tetrahedron having vertices at the four points

$$P:(1,0,1),\ Q:(-1,1,2),\ R:(0,0,2),\ S:(3,1,-1).$$

Hint: volume of tetrahedron $= \frac{1}{6}$(volume of parallelepiped with same 3 coterminous edges)

1D-8 Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

1D-9 Show that the area of a triangle in the xy-plane having vertices at (x_i, y_i), for $i = 1,2,3$, is given by the determinant $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Do this two ways:

a) by relating the area of the triangle to the volume of a certain parallelepiped

b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the 2×2 determinant that would normally be used to calculate the area.

1E. Equations of Lines and Planes

1E-1 Find the equations of the following planes:

a) through $(2, 0, -1)$ and perpendicular to $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
b) through the origin, $(1, 1, 0)$, and $(2, -1, 3)$
c) through $(1, 0, 1)$, $(2, -1, 2)$, $(-1, 3, 2)$
d) through the points on the x, y and z-axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
e) through $(1, 0, 1)$ and $(0, 1, 1)$ and parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

1E-2 Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.

1E-3 Find in parametric form the equations for

a) the line through $(1, 0, -1)$ and parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
b) the line through $(2, -1, -1)$ and perpendicular to the plane $x - y + 2z = 3$
c) all lines passing through $(1, 1, 1)$ and lying in the plane $x + 2y - z = 2$

1E-4 Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

1E-5 The line passing through $(1, 1, -1)$ and perpendicular to the plane $x + 2y - z = 3$ intersects the plane $2x - y + z = 1$ at what point?

1E-6 Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

(Hint: Let $\mathbf{n}$ be the unit normal to the plane. and P be a point on the plane; consider the component of OP in the direction $\mathbf{n}$.)

1E-7* Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

(Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren't, it could be shortened).)

1F. Matrix Algebra

1F-1* Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$. Compute

a) $B + C$, $B - C$, $2B - 3C$.

b) AB, AC, BA, CA, BC^T, CB^T

c) $A(B + C)$, $AB + AC$; $(B + C)A$, $BA + CA$

1F-2* Let A be an arbitrary $m \times n$ matrix, and let I_k be the identity matrix of size k. Verify that $I_m A = A$ and $AI_n = A$.

1F-3 Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1F-4* Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix $AB - BA$:

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$

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1F-5 a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3. b) Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1F-6* Let A, A', B, B' be 2×2 matrices, and O the 2×2 zero matrix. Express in terms of these five matrices the product of the 4×4 matrices $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$.

1F-7* Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Show there are no values of a and b such that $AB - BA = I_2$.

1F-8 a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A?

b)* If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is A?

1F-9 A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that

a) each row of A is a row vector of length 1,

b) two different rows are orthogonal vectors.

1F-10* Suppose A is a 2×2 orthogonal matrix, whose first entry is $a_{11} = \cos\theta$. Fill in the rest of A. (There are four possibilities. Use Exercise 9.)

1F-11* Show that if $A + B$ and AB are defined, then
a) $(A + B)^T = A^T + B^T$, b) $(AB)^T = B^T A^T$.

1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-1* $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$.

1G-2* a) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; b) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

1G-3 $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-4 Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, \quad x_2 + x_3 = y_2 \quad -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i.

1G-5 Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.

1G-6* **Another calculation of the inverse matrix.**

If we know A^{-1}, we can solve the system $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by writing $\mathbf{x} = A^{-1}\mathbf{y}$. But conversely, if we can solve by some other method (elimination, say) for $\mathbf{x}$ in terms of $\mathbf{y}$, getting $\mathbf{x} = B\mathbf{y}$, then the matrix $B = A^{-1}$, and we will have found A^{-1}.

This is a good method if A is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

a) Let $A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$; find A^{-1} by solving $\begin{aligned} -x_1 + x_2 + 3x_3 &= y_1 \\ 2x_2 - x_3 &= y_2 \\ x_3 &= y_3 \end{aligned}$ for the x_i in terms of the y_i (start from the bottom and proceed upwards).

b) Calculate A^{-1} by the method given in the notes.

1G-7* Consider the rotation matrix $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ corresponding to rotation of the x and y axes through the angle θ. Calculate A_θ^{-1} by the adjoint matrix method, and explain why your answer looks the way it does.

1G-8* a) Show: A is an orthogonal matrix (cf. Exercise 1F-9) if and only if $A^{-1} = A^T$.

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if A and B are $n \times n$ orthogonal matrices, so is AB.

1G-9* a) Let A be a 3×3 matrix such that $|A| \neq 0$. The notes construct a right-inverse A^{-1}, that is, a matrix such that $A \cdot A^{-1} = I$. Show that every such matrix A also has a left inverse B (i.e., a matrix such that $BA = I$.)

(Hint: Consider the equation $A^T(A^T)^{-1} = I$; cf. Exercise 1F-11.)

b) Deduce that $B = A^{-1}$ by a one-line argument.

(This shows that the right inverse A^{-1} is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be I also.)

1G-10* Let A and B be two $n \times n$ matrices. Suppose that $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. Show that $B^n = P^{-1}A^nP$. If $B = I_n$, what is A?

1G-11* Repeat Exercise 6a and 6b above, doing it this time for the general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assuming $|A| \neq 0$.

1H. Theorems about Square Systems

1H-1 Use Cramer's rule to solve for x in the following:

$$\text{(a)}\quad \begin{aligned} 3x - y + z &= \ \ 1 \\ -x + 2y + z &= \ \ 2 \\ x - y + z &= -3 \end{aligned}\ , \qquad\qquad \text{(b)}\quad \begin{aligned} x - y + z &= 0 \\ x - z &= 1 \\ -x + y + z &= 2 \end{aligned}.$$

(*We did not cover Cramer's rule in this course.*)

1H-2 Using Cramer's rule, give another proof that if A is an $n\times n$ matrix whose determinant is non-zero, then the equations $A\mathbf{x} = 0$ have only the trivial solution.
(*We did not cover Cramer's rule in this course.*)

1H-3 a) For what c-value(s) will $\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ -x_1 + cx_2 + 2x_3 &= 0 \end{aligned}$ have a non-trivial solution?

b) For what c-value(s) will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} x \\ y \end{pmatrix}$ have a non-trivial solution? (Write it as a system of homogeneous equations.)

c) For each value of c in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)* For each value of c in part (b), find a non-trivial solution to the corresponding system.

1H-4* Find all solutions to the homogeneous system $\begin{aligned} x - 2y + z &= 0 \\ x + y - z &= 0 \\ 3x - 3x + z &= 0 \end{aligned}$; use the method suggested in Exercise 3c above.

1H-5 Suppose that for the system $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Assume that $a_1 \neq 0$. Show that the system is consistent (i.e., has solutions) if and only if $c_2 = \dfrac{a_2}{a_1}c_1$.

1H-6* Suppose $|A| = 0$, and that $\mathbf{x}_1$ is a particular solution of the system $A\mathbf{x} = B$. Show that any other solution $\mathbf{x}_2$ of this system can be written as $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution of the system $A\mathbf{x} = \mathbf{0}$.

1H-7 Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants a and b so that the function

$$f(x) = a\cos x + b\sin x$$

has prescribed values at two given x-values: $f(x_1) = y_1,\ f(x_2) = y_2$.

a) Show this is possible in one and only one way, if we assume that $x_2 \neq x_1 + n\pi$, for every integer n.

b) If $x_2 = x_1 + n\pi$ for some integer n, when can a and b be found?

1H-8* The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}, \qquad (a, b, r_1, r_2 \text{ given; } c, d \text{ to be found});$$

what are the linear equations which determine the constants c and d? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots r_i, $i = 1, 2, 3$. Evaluate the determinant by using column operations to get zeros in the top row.)

1I. Vector Functions and Parametric Equations

1I-1 The point P moves with constant speed v in the direction of the constant vector $a\,\mathbf{i} + b\,\mathbf{j}$. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $\mathbf{r}(t)$?

1I-2 A point moves *clockwise* with constant angular velocity ω on the circle of radius a centered at the origin. What is its position vector function $\mathbf{r}(t)$, if at time $t = 0$ it is at

(a) $(a, 0)$ (b) $(0, a)$

1I-3 Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.

a) $\mathbf{r} = 2\cos^2 t\,\mathbf{i} + \sin^2 t\,\mathbf{j}$ b) $\mathbf{r} = \cos 2t\,\mathbf{i} + \cos t\,\mathbf{j}$ c) $\mathbf{r} = (t^2+1)\,\mathbf{i} + t^3\,\mathbf{j}$
d) $\mathbf{r} = \tan t\,\mathbf{i} + \sec t\,\mathbf{j}$

1I-4 A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O, and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P.

(Use vectors; express the position vector OP as a vector function of one variable.)

1I-5 A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.

(Use vectors; express the position vector OP as a vector function of one variable.)

1I-6 A bow-and-arrow hunter walks toward the origin along the positive x-axis, with unit speed; at time 0 he is at $x = 10$. His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity $\sqrt{5}$ in the first quadrant along the line $y = 2x$; at time 0 it is at the origin.

a) Write down the vector function $\mathbf{A}(t)$ for the arrow at time t.

b) The hunter shoots (and misses) when closest to the rabbit; when is that?

1I-7 The cycloid is the curve traced out by a fixed point P on a circle of radius a which rolls along the x-axis in the positive direction, starting when P is at the origin O. Find the vector function OP; use as variable the angle θ through which the circle has rolled.

(Hint: begin by expressing OP as the sum of three simpler vector functions.)

1J. Differentiation of Vector Functions

1J-1 1. For each of the following vector functions of time, calculate the velocity, speed $|ds/dt|$, unit tangent vector (in the direction of velocity), and acceleration.

a) $e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ b) $t^2\,\mathbf{i} + t^3\,\mathbf{j}$ c) $(1-2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2+2t^2)\,\mathbf{k}$

1J-2 Let $OP = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$ be the position vector for a motion.

a) Calculate $\mathbf{v}$, $|ds/dt|$, and $\mathbf{T}$.

b) At what point in the speed greatest? smallest?

c) Find the xy-equation of the curve along which the point P is moving, and describe it geometrically.

1J-3 Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt}\,\mathbf{r}\cdot\mathbf{s} = \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\ ,$$

by calculating with components, letting $\mathbf{r} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{s} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$.

1J-4 Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\,.$$

Show that the velocity vector $\mathbf{v}$ is always perpendicular to $\mathbf{r}$ two different ways:

a) using the x, y, z-coordinates

b) without coordinates (use the formula in **1J-3**, which is valid also in space).

c) Prove the converse: if $\mathbf{r}$ and $\mathbf{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin.

1J-5 a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**.)

b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point P moves with constant speed.

1J-6 For the helical motion $r(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$,

a) calculate $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, $|ds/dt|$

b) show that $\mathbf{v}$ and $\mathbf{a}$ are perpendicular; explain using **1J-5**

1J-7 a) Suppose you have a differentiable vector function $\mathbf{r}(t)$. How can you tell if the parameter t is the arclength s (measured from some point in the direction of increasing t) without actually having to calculate s explicitly?

b) How should a be chosen so that t is the arclength if $\mathbf{r}(t) = (x_0 + at)\,\mathbf{i} + (y_0 + at)\,\mathbf{j}$?

c) How should a and b be chosen so that t is the arclength in the helical motion described in Exercise **1J-6**?

1J-8 a) Prove the formula $\dfrac{d}{dt}\,u(t)\mathbf{r}(t) \;=\; \dfrac{du}{dt}\,\mathbf{r}(t) + u(t)\dfrac{d\mathbf{r}}{dt}$.

(You may assume the vectors are in the plane; calculate with the components.)

b) Let $\mathbf{r}(t) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j}$, the exponential spiral. Use part (a) to find the speed of this motion.

1J-9 A point P is moving in space, with position vector

$$\mathbf{r} = OP = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k}$$

a) Show it moves on the surface of a sphere.
b) Show its speed is constant.
c) Show the acceleration is directed toward the origin.
d) Show it moves in a plane through the origin.
e) Describe the path of the point.

1J-10 The **positive curvature** κ of the vector function $\mathbf{r}(t)$ is defined by $\kappa \;=\; \left|\dfrac{d\mathbf{T}}{ds}\right|$.

a) Show that the helix of **1J-6** has constant curvature. (It is not necessary to calculate s explicitly; calculate $d\mathbf{T}/dt$ instead and relate it to κ by using the chain rule.)

b) What is this curvature if the helix is reduced to a circle in the xy-plane?

1K. Kepler's Second Law

1K-1 (Same as 1J-3). Prove the product rule for differentiating the dot product of two plane vectors: do the calculation using an $\mathbf{i}\,\mathbf{j}$-coordinate system.
(Let $\mathbf{r}(t) = x_1(t)\,\mathbf{i} + y_1(t)\,\mathbf{j}$ and $\mathbf{s}(t) = x_2(t)\,\mathbf{i} + y_2(t)\,\mathbf{j}$.)

1K-2 Let $\mathbf{s}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{s}}{dt} = \mathbf{0} \quad\Rightarrow\quad \mathbf{s}(t) = \mathbf{K}, \quad \text{where } \mathbf{K} \text{ is a constant vector.}$$

1K-3 In our proof that Kepler's second law is equivalent to the force being central, used the following steps to show the second law implies a central force. Kepler's second law says the motion is in a plane and

$$2\frac{dA}{dt} = |\mathbf{r}\times\mathbf{v}| \text{ is constant.} \tag{2}$$

This implies $\mathbf{r}\times\mathbf{v}$ is constant. So,

$$0 = \frac{d}{dt}(\mathbf{r}\times\mathbf{v}) \;=\; \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = \mathbf{r}\times\mathbf{a}.$$

This implies $\mathbf{a}$ and $\mathbf{r}$ are parallel, i.e. the force is central.

Reverse these steps to prove the converse: for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate. You will need the statement in exercise 1K-2.