18.02 Practice Exam 1 – Solutions

Problem 1.

a) $\overrightarrow{OQ} = \hat{\imath} + \hat{\jmath} + \hat{k}$; $\overrightarrow{OR} = \dfrac{1}{2}\hat{\imath} + \hat{\jmath} + \dfrac{1}{2}\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{OQ}\cdot\overrightarrow{OR}}{|\overrightarrow{OQ}|\,|\overrightarrow{OR}|} = \dfrac{\langle 1,1,1\rangle\cdot\langle \frac{1}{2},1,\frac{1}{2}\rangle}{\sqrt{3}\sqrt{\frac{3}{2}}} = \dfrac{2\sqrt{2}}{3}$.

Problem 2.

Velocity: $\vec{V} = \dfrac{d\vec{R}}{dt} = \langle -3\sin t, 3\cos t, 1\rangle$. Speed: $|\vec{V}| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$.

Problem 3.

a) Minors: $\begin{bmatrix} 1 & 1 & 2 \\ -2 & -2 & -2 \\ -3 & -5 & -6 \end{bmatrix}$. Cofactors: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 2 \\ -3 & 5 & -6 \end{bmatrix}$. Inverse: $\dfrac{1}{2}\begin{bmatrix} 1 & \boxed{2} & \boxed{-3} \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$.

b) $X = A^{-1}B = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$.

c) Not invertible is equivalent to determinant is 0:

$$\begin{vmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (1) - 3(1) + c(2) = 2c - 2 = 0 \Leftrightarrow c = 1.$$

One solution to the equation comes from the cross product of 2 of the rows:

$$\langle 1,3,1\rangle \times \langle 2,0,-1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \hat{\imath}(-3) - \hat{\jmath}(-3) + \hat{k}(-6) = \langle -3, 3, -6\rangle.$$

(Any non-zero multiple of this is also correct.)

Problem 4.

Q = top of the ladder: $\overrightarrow{OQ} = \langle 0, L\sin\theta\rangle$; R = bottom of the ladder: $\overrightarrow{OR} = \langle -L\cos\theta, 0\rangle$.
Midpoint: $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OR}) = \langle -\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\rangle$.
Parametric equations: $x = -\frac{L}{2}\cos\theta$, $y = \frac{L}{2}\sin\theta$.

Problem 5.

a) $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Area $= \frac{1}{2}|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| = \frac{1}{2}\sqrt{6}$.

b) Normal vector: $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$. Equation: $x + y + 2z = 3$.

c) Parametric equations for the line: $x = -1 + t$, $y = t$, $z = t$.
Substituting: $-1 + 4t = 3$, $t = 1$, intersection point $(0, 1, 1)$.

Problem 6.

a) $\dfrac{d}{dt}(\vec{R}\cdot\vec{R}) = \vec{V}\cdot\vec{R} + \vec{R}\cdot\vec{V} = 2\vec{R}\cdot\vec{V}$.

b) Assume $|\vec{R}|$ is constant: then $\dfrac{d}{dt}(\vec{R}\cdot\vec{R}) = 2\vec{R}\cdot\vec{V} = 0$, i.e. $\vec{R}\perp\vec{V}$.

c) $\vec{R}\cdot\vec{V} = 0$, so $\dfrac{d}{dt}(\vec{R}\cdot\vec{V}) = \vec{V}\cdot\vec{V} + \vec{R}\cdot\vec{A} = 0$. Therefore $\vec{R}\cdot\vec{A} = -|\vec{V}|^2$.