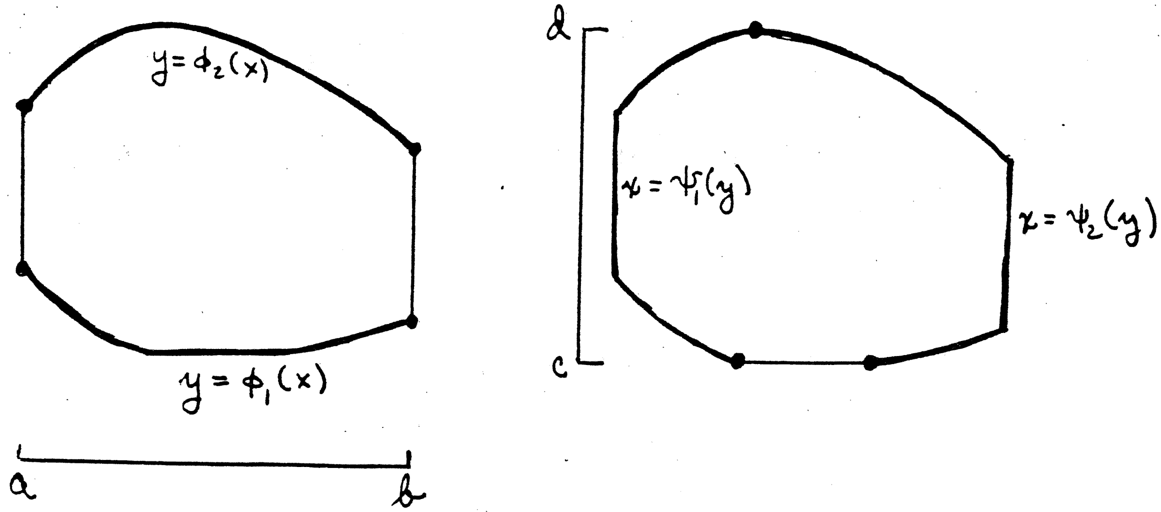
GREEN'S THEOREM AND ITS APPLICATIONS

The discussion in 11.19 - 11.27 of Apostol is not complete nor entirely rigorous, as the author himself points out. We give here a rigorous treatment.

§11.19 Green's Theorem in the Plane

We already know what is meant by saying that a region in the plane is of Type I or of Type II or that it is of both types simultaneously. Apostol proves Green's Theorem for a region that is of both types. Such a region R can be described in two different ways, as follows:



R: $a < x < b$

$\phi_1(x) < y < \phi_2(x)$

R: $c < y < d$

$\psi_1(y) < x < \psi_2(y)$

The author's proof is complete and rigorous except for one gap, which arises from his use of the intuitive notion of "counter-clockwise".

Specifically, what he does is the following: For the first part of the proof he orients the boundary C of R as follows:

(*) By increasing x, on the curve $y = \phi_1(x)$;
By increasing y, on the line segment $x = b$;
By decreasing x, on the curve $y = \phi_2(x)$; and
By decreasing y, on the line segment $x = a$.

Then in the second part of the proof, he orients C as follows:

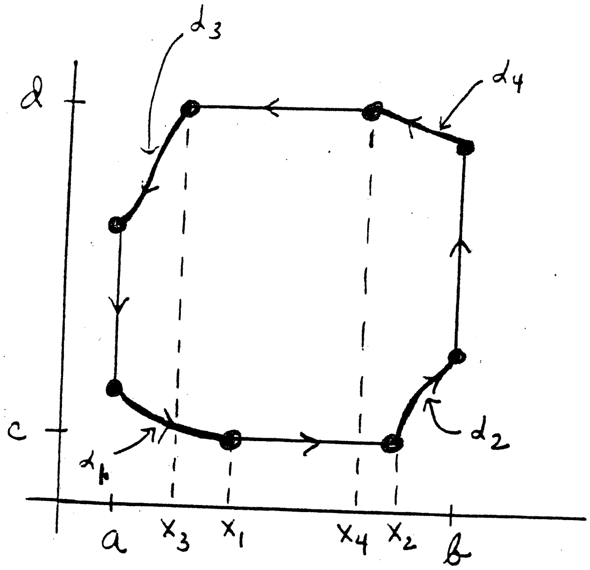
(**) By decreasing y, on the curve $x = \psi_1(y)$;
By increasing x, on the line segment $y = c$;
By increasing y, on the curve $x = \psi_2(y)$; and
By decreasing x, on the line segment $y = d$.

(The latter line segment collapses to a single point in the preceding figure.)

The crucial question is: <u>How does one know these two orientations of C are the same</u>?

One can in fact see that these two orientations are the same, by simply analyzing a bit more carefully what one means by a region of Types I and II.

Specifically, such a region can be described by four monotonic functions:



$y = \alpha_1(x);\ a \leq x \leq x_1,$

$y = \alpha_2(x);\ x_2 \leq x \leq b,$

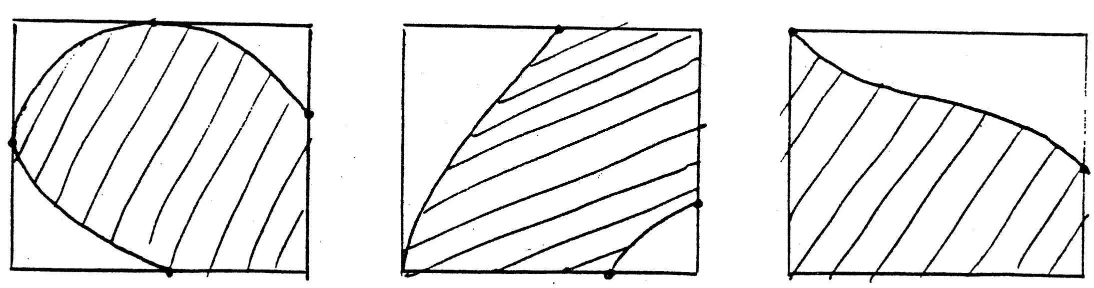
$y = \alpha_3(x);\ a \leq x \leq x_3,$

$y = \alpha_4(x);\ x_4 \leq x \leq b,$

where α_1 and α_4 are strictly decreasing and α_2 and α_3 are strictly increasing. We require that

$a \leq x_1 \leq x_2 \leq b$ and $a \leq x_3 \leq x_4 \leq b$; and that $\alpha_1(a) \leq \alpha_3(a)$ and $\alpha_1(x_1) = \alpha_2(x_2)$ and $\alpha_3(x_3) = \alpha_4(x_4)$ and $\alpha_2(b) \leq \alpha_4(b)$, as in the picture.

[Some or all of the α_i can be missing, of course. Here are pictures of typical such regions:]



The curves α_1 and α_2, along with the line segment $y = c$, are used to define the curve $y = \phi_1(x)$ that bounds the region on the bottom. Similarly, α_3 and α_4 and $y = d$ define the curve $y = \phi_2(x)$ that bounds the region on the top.

Similarly, the inverse functions to α_1 and α_3, along with $x = a$, combine to define the curve $x = \psi_1(y)$ that bounds the region on the left; and the inverse functions to α_2 and α_4, along with $x = b$, define the curve $x = \psi_2(y)$.

Now one can choose a direction on the bounding curve C by simply directing each of these eight curves as indicated in the figure, and check that this is the same as the directions specified in (*) and (**). [Formally, one directs these curves as follows:

increasing x = decreasing y	on $y = \alpha_1(x)$
increasing x	on $y = c$
increasing x = increasing y	on $y = \alpha_2(x)$
increasing y	on $x = b$
decreasing x = increasing y	on $y = \alpha_4(x)$
decreasing x	on $y = d$
decreasing x = decreasing y	on $y = \alpha_3(x)$
decreasing y	on $x = a$.]

We make the following definition:

<u>Definition</u>. Let R be an open set in the plane bounded by a simple closed piecewise-differentiable curve C. We say that R is a <u>Green's region</u> if it is possible to choose a direction on C so that the equation

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

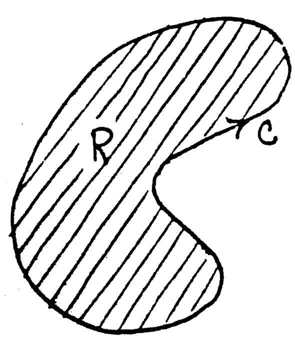
holds for every continuously differentiable vector field $P(x,y)\vec{i} + Q(x,y)\vec{j}$ that is defined in an open set containing R and C.

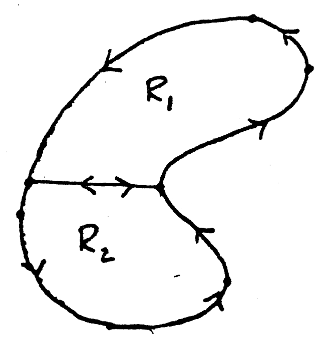
The direction on C that makes this equation correct is called the counterclockwise direction, or the counterclockwise orientation, of C.

In these terms, Theorem 11.10 of Apostol can be restated as follows:

Theorem 1. Let R be bounded by a simple closed piecewise-differentiable curve. If R is of Types I and II, then R is a Green's region.

As the following figure illustrates, almost any region R you are likely to draw can be shown to be a Green's region by repeated application of this theorem. In such a case, the counterclockwise direction on C is by definition the one for which Green's theorem holds. For example, the region R is a Green's region, and the counterclockwise orientation of its boundary C is as indicated. The figure on the right indicates the proof that it is a Green's region; each of R_1 and R_2 is of Types I and II.



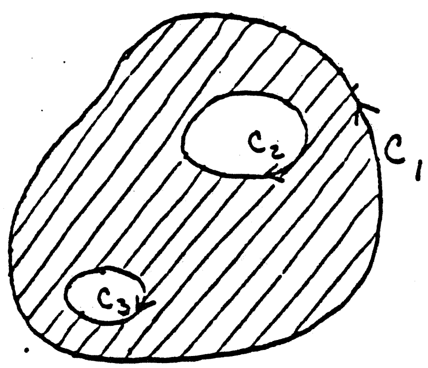


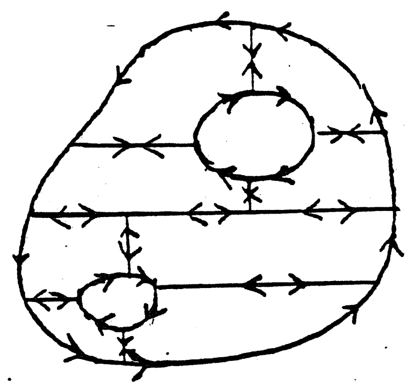
Definition. Let R be a bounded region in the plane whose boundary is the union of the disjoint piecewise-differentiable simple closed curves $C_1, \ldots, C_n$. We call R a generalized Green's region if it is possible to direct the curves $C_1, \ldots, C_n$ so that the equation

$$\int_{C_1+C_2+\cdots+C_n} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

holds for every continuously differentiable vector field $P\vec{i} + Q\vec{j}$ defined in an open set about R and C.

Once again, every such region you are likely to draw can be shown to be a generalized Green's region by several applications of Theorem 1. For example, the region R pictured is a generalized Green's region if its boundary is directed as indicated. The proof is indicated in the figure on the right. One applies Theorem 1 to each of the 8 regions pictured and adds the results together.



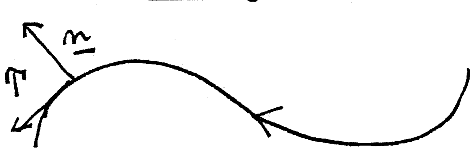


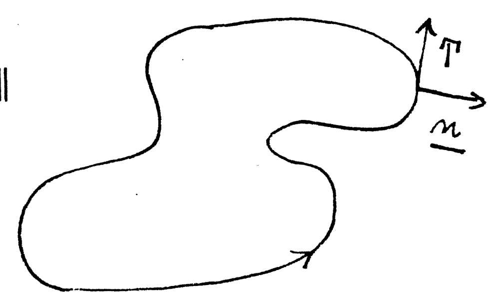
Exercises.

Definition. Let C be a piecewise–differentiable curve in the plane parametrized by the function $\underline{\alpha}(t) = (x(t),y(t))$. The vector $T = (x'(t),y'(t))/\|\underline{\alpha}'(t)\|$ is the unit tangent vector to C. The vector

$$\underline{n} = (y'(t),-x'(t))/\|\underline{\alpha}'(t)\|$$

is called the unit negative normal to C.





If C is a simple closed curve oriented counterclockwise, then $\underline{n}$ is the "outward normal" to C.

① If $\underline{f} = P\vec{i} + Q\vec{j}$ is a continuously differentiable vector field defined in an open set containing C, then the integral $\int_C (\underline{f}\cdot\underline{n})dS$ is well–defined; show that it equals the line integral

$$\int_C -Q\,dx + P\,dy.$$

② Show that if C bounds a region R that is a Green's region, then $\oint_C (\underline{f}\cdot\underline{n})dS = \iint_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dx\,dy$.

[Remark. If $\underline{f}$ is the velocity of a fluid, then $\int_C (\underline{f}\cdot\underline{n})dS$ is the area of fluid flowing outward through C in unit time. Thus $\partial P/\partial x + \partial Q/\partial y$ measures the rate of expansion of the fluid, per unit area. It is called the divergence of $\underline{f}$.]

Definition. Let ϕ be a scalar field (continuously differentiable) defined on C. If $\underline{x}$ is a point of C, then $\phi'(\underline{x};\underline{n})$ is the directional derivative of ϕ in the direction of $\underline{n}$. It is equal to $\vec{\nabla}\phi(\underline{x})\cdot\underline{n}$, of course. Physicists and engineers use the (lousy) notation $\frac{\partial\phi}{\partial n}$ to denote this directional derivative.

③ Let R be a Green's region bounded by C. Let f and g be scalar fields (with continuous first and second partials) in an open set about R and C.

(a) Show $\oint_C \frac{\partial g}{\partial n}\,ds = \iint_R \nabla^2 g\,dx\,dy$

where $\nabla^2 g = \partial^2 g/\partial x^2 + \partial^2 g/\partial y^2$.

(b) Show

$$\oint_C f\,\frac{\partial g}{\partial n}\,ds = \iint_R (f\nabla^2 g + \vec{\nabla} f\cdot\vec{\nabla} g)dx\,dy.$$

(c) If $\nabla^2 f = 0 = \nabla^2 g$, show

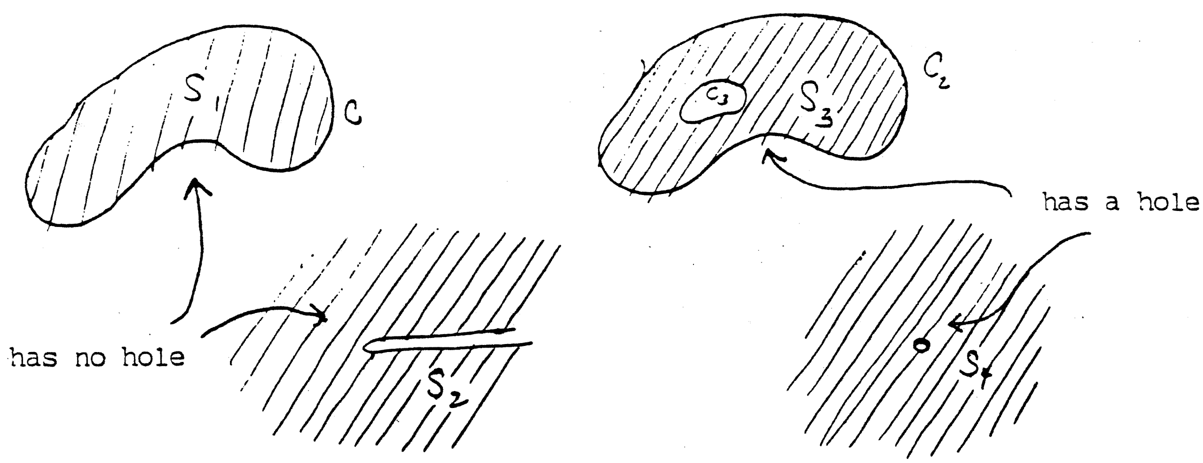
$$\oint_C f\,\frac{\partial g}{\partial n}\,ds = \oint_C g\,\frac{\partial f}{\partial n}.$$

These equations are important in applied math and classical physics. A function f with $\nabla^2 f = 0$ is said to be harmonic. Such functions arise in physics: In a region free of charge, electrostatic potential is harmonic; for a body in temperature equilibrium, the temperature function is harmonic.

Conditions Under Which $P\vec{i} + Q\vec{j}$ is a Gradient.

Let $\underline{f} = P\vec{i} + Q\vec{j}$ be a continuously differentiable vector field defined on an open set S in the plane, such that $\partial P/\partial y = \partial Q/\partial x$ on S. In general, we know that $\underline{f}$ need not be a gradient on S. We do know that $\underline{f}$ will be a gradient if S is convex (or even if S is star-convex). We seek to extend this result to a more general class of plane sets.

This more general class may be informally described as consisting of those regions in the plane that have no "holes". For example, the region S_1 inside a simple closed curve C_1 has



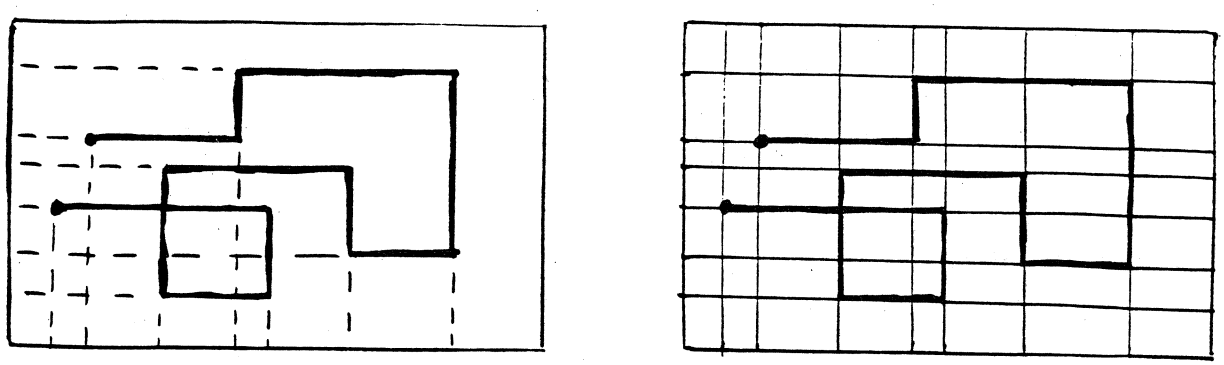
no holes, nor does the region S_2 obtained from the plane by deleting the non-negative x-axis. On the other hand, the region S_3 consisting of the points inside C_2 and outside C_3 has a hole, and so does the region S_4 obtained from the plane by deleting the origin.

Needless to say, we must make this condition more precise if we are to prove a theorem about it. This task turns out to be surprisingly difficult.

We begin by proving some facts about the geometry of the plane.

<u>Definition</u>. A <u>stairstep curve</u> C in the plane is a curve that is made up of finitely many horizontal and vertical line segments.

For such a curve C, we can choose a rectangle Q whose interior contains C. Then by using the coordinates of the end points of the line segments of the curve C as partition points, we can construct a partition of Q such that C is made up entirely of edges of subrectangles of this partition. This process is illustrated in the following figure:

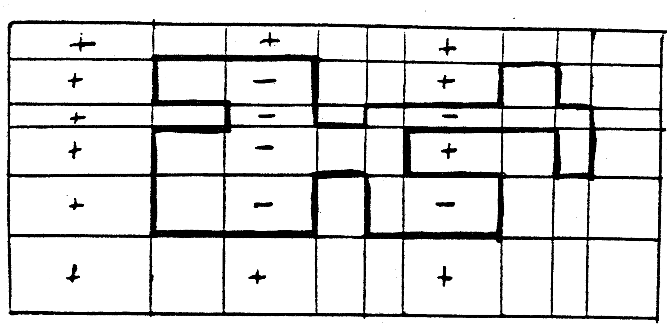


Theorem 2. (The Jordan curve theorem for stairstep curves). Let C be a simple closed stairstep curve in the plane. Then the complement of C can be written as the union of two disjoint open sets. One of these sets is bounded and the other is unbounded. Each of them has C as its boundary.

Proof. Choose a rectangle Q whose interior contains C, and a partition of Q, say $x_0 < x_1 < \ldots < x_n$ and $y_0 < y_1 < \ldots < y_m$, such that C is made up of edges of subrectangles of this partition.

Step 1. We begin by marking each of the rectangles in the partition + or - by the following rule:

Consider the rectangles in the $i^{\underline{th}}$ "column" beginning with the bottom one. Mark the bottom one with +. In general, if a given rectangle is marked with + or -, mark the one just above it with the same sign if their common edge does not lie in C, and with the opposite sign if this edge does lie in C. Repeat this process for each column of rectangles. In the following figure, we have marked the rectangles in columns 1,3, and 6, to illustrate the process.

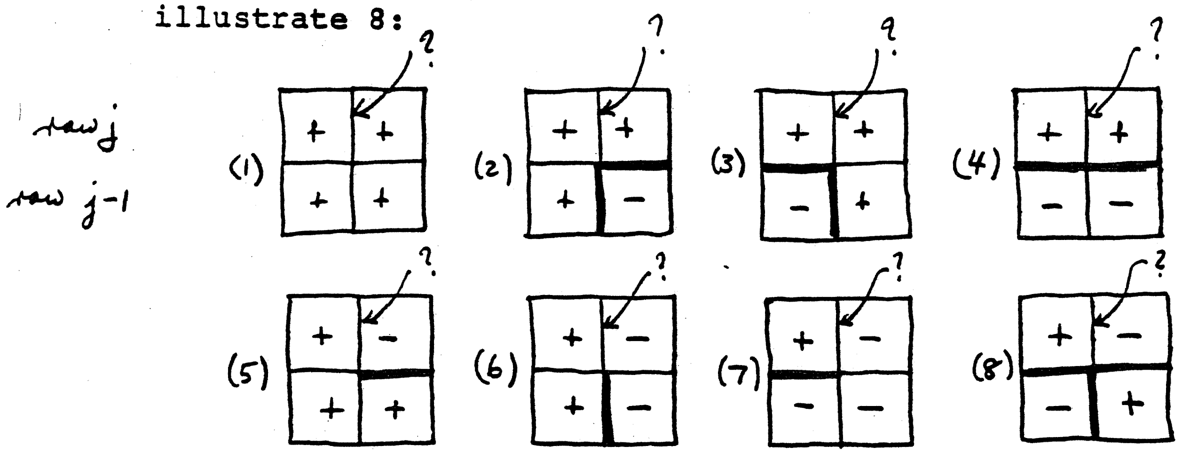


Note that the rectangles in the bottom row are always marked +, and so are those in the first and last columns, (since C does not touch the boundary of Q).

Step 2. We prove the following: *If two subrectangles of the partition have an edge in common, then they have opposite signs if that edge is in C, and they have the same sign if that edge is not in C.*

This result holds by construction for the horizontal edges. We prove it holds for the vertical edges, by induction.

It is true for each of the lowest vertical edges, those of the form $x_i \times [y_0, y_1]$. (For no such edge is in C, and the bottom rectangles are all marked +.) Supposing now it is true for the rectangles in row j - 1, we prove it true for rectangles in row j. There are 16 cases to consider (!), of which we illustrate 8:



(The other eight are obtained from these by changing all the signs.) We know in each case, by construction, whether the two horizontal edges are in C, and we know from the induction hypothesis whether the lower vectical edge is in C. Those edges that we know are in C are marked heavily in the figure. We seek to determine whether the upper vertical edge (marked "?") is in C or not. We use the fact that C is a simple closed curve, which implies in particular that each vertex in C lies on exactly two edges in C. In case (1), this means that the upper vertical edge is not in C, for otherwise the middle vertex would be on only one edge of C. Similarly, in cases (2), (3), and (4), the upper vertical edge is not in C, for otherwise the middle vertex would lie on three edges of C.

Similar reasoning shows that in cases (5), (6), and (7) the upper vertical edge must lie in C, and it shows that case (8) cannot occur.

Thus Step 2 is proved in these 8 cases. The other 8 are symmetric to these, so the same proof applies.

<u>Step</u> 3. It follows from Step 2 that the top row of rectangles is marked +, since the upper left and upper right rectangles are marked +, and C does not touch the boundary of Q.

<u>Step</u> 4. We divide all of the complement of C into two sets U and V as follows. Into U we put the interiors of all rectangles marked −, and into V we put the interiors of all rectangles marked +. We also put into V all points of the plane lying outside and on the boundary of Q. We still have to decide where to put the edges and vertices of the partition that do not lie in C.

Consider first an edge lying interior to Q. If it does not lie in the curve C, then both its adjacent rectangles lie in U or both lie in V (by Step 2); put this (open) edge in U or in V accordingly. Finally, consider a vertex v that lies interior to Q. If it is not on the curve C, then case (1) of the preceding eight cases (or the case with opposite signs) holds. Then all four of the adjacent rectangles are in U or all four are in V; put v into U or V accordingly.

It is immediately clear from the construction that U and V are open sets; any point of U or V (whether it is interior to a subrectangle, or on an edge, or is a vertex) lies in an open ball contained entirely in U or V. It is also immediate that U is bounded and V is unbounded. Furthermore, C is the common boundary of U and V, because for each edge lying in C, one of the adjacent rectangles is marked + and the other is marked −, by Step 2. □

Definition. Let C be a simple closed stairstep curve in the plane. The bounded open set U constructed in the preceding proof is called the inner region of C, or the region inside C.

It is true that U and V are connected, but the proof is difficult. We shall not need this fact.

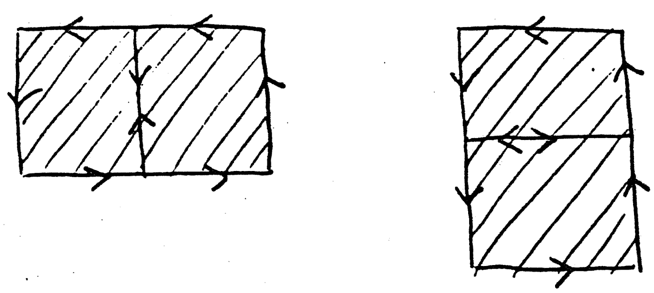
Definition. Let S be an open connected set in the plane. Then S is called simply connected, if, for every simple closed stairstep curve C which lies in S, the inner region of C is also a subset of S.

Theorem 3. If U is the region inside a simple closed stairstep curve C, then U is a Green's region.

Proof. Choose a partition of a rectangle Q enclosing U such that C consists entirely of edges of subrectangles of the partition. For each subrectangle Q_{ij} of this partition lying in U, it is true that

$$\int_{C_{ij}} Pdx + Qdy = \iint_{Q_{ij}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

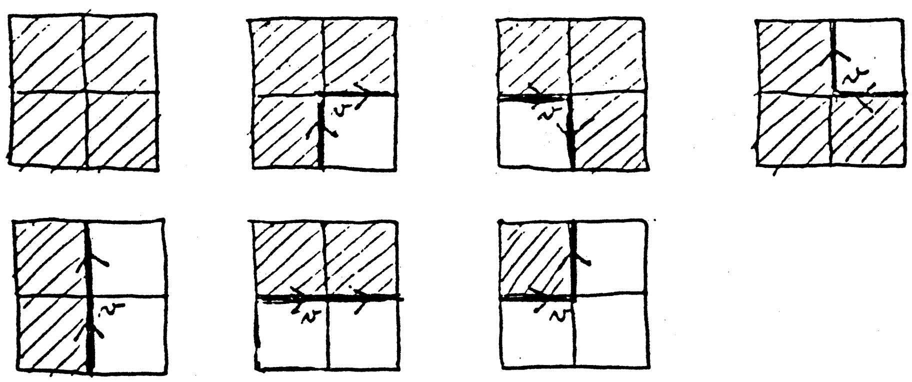
if C_{ij} is the boundary of Q_{ij}, traversed in a counterclockwise direction. (For Q_{ij} is a type I-II region). Now each edge of the partition lying in C appears in only one of these curves C_{ij}, and each edge of the partition not lying in C appears in either none of the C_{ij}, or it appears in two of the C_{ij} with oppositely directed arrows, as indicated:



If we sum over all subrectangles Q_{ij} in U, we thus obtain the equation

$$\int_{(\text{line segments in } C)} Pdx + Qdy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

The only question is whether the directions we have thus given to the line segments lying in C combine to give an orientation of C. That they do is proved by examining the possible cases. Seven of them are as follows; the other seven are opposite to them.



These diagrams show that for each vertex v of the partition such that v is on the curve C, v is the initial point of one of the two line segments of C touching it, and the final point of the other. □

Theorem 4. Let S be an open set in the plane such that every pair of points of S can be joined by a stairstep curve in S. Let

$$\underline{f}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

be a vector field that is continuously differentiable in S, such that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

on all of S. (a) If S is simply connected, then f is a gradient in S. (b) If S is not simply connected, then f may or may not be a gradient in S.

Proof. The proof of (b) is left as an exercise. We prove (a) here. Assume that S is simply connected.

Step 1. We show that

$$\oint_C Pdx + Qdy = 0$$

for every simple closed stairstep curve C lying in S.

We know that the region U inside C is a Green's region. We also know that the region U lies entirely within S. (For if there were a point p of U that is not in S, then C encircles a point p not in S, so that S has a hole at p. This contradicts the fact that S is simply connected.) Therefore the equation $\partial Q/\partial x = \partial P/\partial y$ holds on all of U; we therefore conclude that

$$\oint_C Pdx + Qdy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = 0,$$

for some orientation of C (and hence for both orientations of C).

Step 2. We show that if

$$\oint_C Pdx + Qdy = 0$$

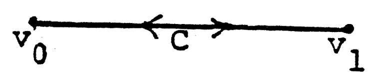
for every simple closed stairstep curve in S, then the same equation holds for every stairstep curve in S.

Assume C consists of the edges of subrectangles in a partition of some rectangle that contains C, as usual.

We proceed by induction on the number of vertices on the curve C. Consider the vertices of C in order:

$$v_0, v_1, \ldots, v_n, v_0.$$

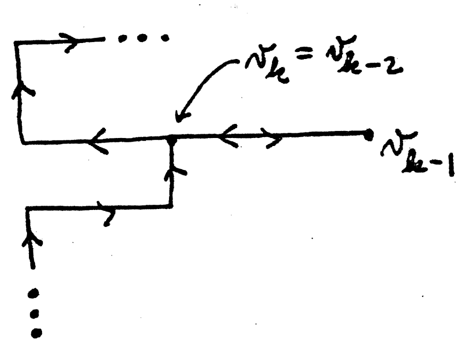
Now C cannot have only one vertex. If it has only two, then C is a path going from v_0 to v_1 and then back to v_0. The line integral vanishes



in this case.

Now suppose the theorem true for curves with fewer than n vertices. Let C have n vertices. If C is a simple curve, we are through. Otherwise, let v_k be the first vertex in this sequence that equals some earlier vertex v_i for $i < k$. We cannot have $v_k = v_{k-1}$, for then $v_{k-1}v_k$ would not be a line segment.

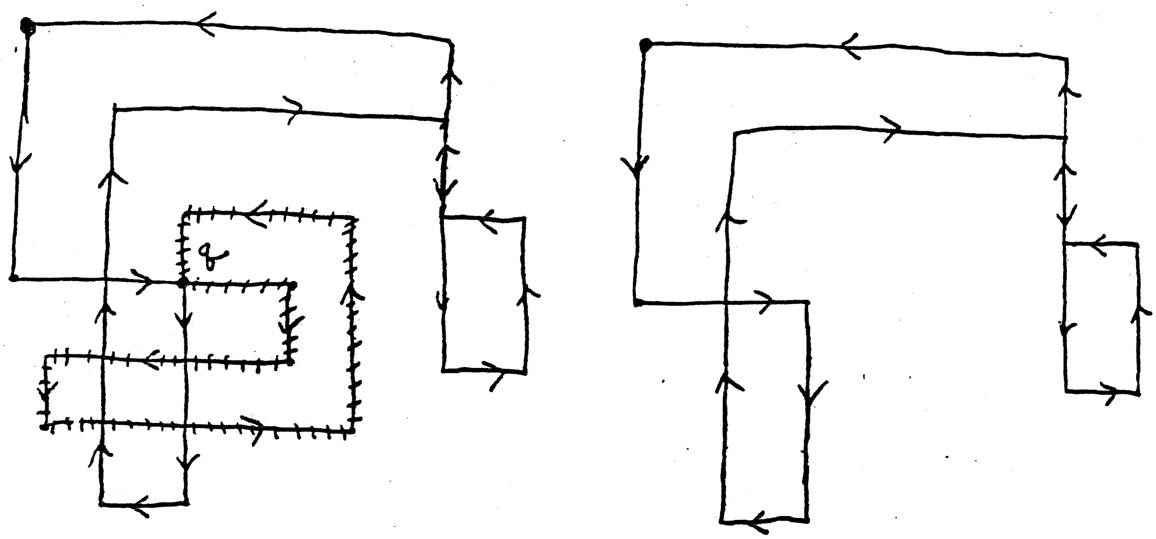
If $v_k = v_{k-2}$, then the curve contains the line segment $v_{k-2}v_{k-1}$, followed by the same line segment in reverse order. Then the integral from v_{k-2} to v_{k-1} and the integral from



v_{k-1} to v_k are negatives of each other. We can delete v_{k-1} from the sequence of vertices without changing the value of the integral. We have a closed curve remaining with fewer line segments than before, and the induction hypothesis applies.

If $i < k-2$, then we can consider the closed curve with vertices $v_i, v_{i+1}, \ldots, v_k$. This is a <u>simple</u> closed curve, since all its vertices are distinct, so the integral around it is zero, by Step 1. Therefore the value of the integral $\int_C Pdx + Qdy$ is not changed if we delete this part of C, i.e., if we delete the vertices $v_i, \ldots, v_{k-1}$ from the sequence. Then the induction hypothesis applies.

<u>Example</u>. In the following case,



the first vertex at which the curve touches a vertex already touched is the point q. One considers the simple closed cross-hatched curve, the integral around which is zero. Deleting this curve, one has a curve remaining consisting of fewer line segments. You can continue the process until you have a simple closed curve remaining.

Step 3. We show that if C_1 and C_2 are any two stairstep curves in S from p to q, then

$$\int_{C_1} Pdx + Qdy = \int_{C_2} Pdx + Qdy.$$

This follows by the usual argument. If $-C_2$ denotes C_2 with the reversed direction, then $C = C_1 + (-C_2)$ in a closed stairstep curve. We have

$$\int_{C_1} - \int_{C_2} = \int_{C_1} + \int_{-C_2} = \oint_C \ .$$

This last integral vanishes, by Step 2.

Step 4. Now we prove the theorem. Let $\underline{a}$ be a fixed point of S, and define

$$\phi(\underline{x}) = \int_{C(\underline{x})} Pdx + Qdy.$$

where $C(\underline{x})$ is any stairstep curve in S from $\underline{a}$ to $\underline{x}$. There always exists such a stairstep curve (by hypothesis), and the value of the line integral is independent of the choice of the curve (by Step 3). It remains to show that

$$\partial\phi/\partial x = P \text{ and } \partial\phi/\partial y = Q.$$

We proved this once before under the assumption that $C(\underline{x})$ was an arbitrary piecewise smooth curve. But the proof works just as well if we require $C(\underline{x})$ to be a stairstep curve. To compute $\partial\phi/\partial x$, we first computed $[\phi(x+h,y) - \phi(x,y)]/h$. We computed $\phi(x,y)$ by choosing a curve C_1 from $\underline{a}$ to (x,y), and integrated along C_1. We computed $\phi(x+h,y)$ by choosing this same curve C_1 plus the straight line C_2 from (x,y) to $(x+h,y)$. In the present case, we have required C_1 to be a stairstep curve. Then we note that if C_1 is a stairstep curve, $C_1 + C_2$ is also a stairstep curve. Therefore the earlier proof goes through without change. □

Remark. It is a fact that if two pair of points of S can be joined by some path in S, then they can be joined by a stairstep path. (We shall not bother to prove this fact.) It follows that the hypothesis of the preceding theorem is merely that S be connected and simply connected.

Exercises

1. Let S be the punctured plane, i.e., the plane with the origin deleted. Show that the vector fields

$$\underline{f} = \frac{x\vec{i} + y\vec{j}}{x^2 + y^2} \qquad \underline{g} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$$

satisfy the condition $\partial P/\partial y = \partial Q/\partial x$.

(a) Show that $\underline{f}$ is a gradient in S. [Hint: First find ϕ so that $\partial\phi/\partial x = x/(x^2 + y^2)$.] (b) Show that $\underline{g}$ is not a gradient in S. [Hint: Compute $\int_C \underline{g} \cdot d\underline{\alpha}$ where C is the unit circle centered at the origin.]

2. Prove the following:

Theorem 5. Let C_1 be a simple closed stairstep curve in the plane. Let C_2 be a simple closed stairstep curve that is contained in the inner region of C_1. Show that the region consisting of those points that are in the inner region of C_1 and are not on C_2 nor in the inner region of C_2 is a generalized Green's region, bounded by C_1 and C_2.

[Hint: Follow the pattern of the proof of Theorem 3.]

3. Let $\underline{g}$ be the vector field of Exercise 1. Let C be any simple closed stairstep curve whose inner region contains $\underline{0}$. Show that $\int_C \underline{f}\cdot d\underline{r} \neq 0$. [Hint: Show this inequality holds if C is the boundary of a rectangle. Then apply Theorem 5.]

*4. Even if the region S is not simply connected, one can usually determine whether a given vector field equals a gradient field in S. Here is one example, where the region S is the punctured plane.

Theorem 6. *Suppose that* $\underline{f} = P\underline{i} + Q\underline{j}$ *is continuously differentiable and*

$$\partial Q/\partial x = \partial P/\partial y$$

in the punctured plane. Let R *be a fixed rectangle enclosing the origin; orient* Bd R *counterclockwise; let*

$$A = \int_{\text{Bd } R} P\,dx + Q\,dy.$$

(a) *If* C *is any simple closed stairstep curve not touching the origin, then*

$$\int_C P\,dx + Q\,dy$$

either equals $\pm A$ (if the origin is in the inner region of C) *or* 0 (otherwise).

(b) If $A = 0$, then $\underline{f}$ equals a gradient field in the punctured plane. [Hint: Imitate the proof of Theorem 4.]

(c) If $A \neq 0$, then $\underline{f}$ differs from a gradient field by a constant multiple of the vector field

$$\underline{g}(\underline{x}) = (-y\underline{i} + x\underline{j})/(x^2 + y^2).$$

That is, there is a constant c such that $\underline{f} + c\underline{g}$ equals a gradient field in the punctured plane. (Indeed, $c = -A/2\pi$.)

$$\frac{\partial Q_1}{\partial u} = \left(\frac{\partial Q}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Q}{\partial y}\frac{\partial Y}{\partial u}\right)\frac{\partial Y}{\partial v} + Q\frac{\partial^2 Y}{\partial u \partial v}$$

$$\frac{\partial P_1}{\partial v} = \left(\frac{\partial Q}{\partial x}\frac{\partial X}{\partial v} + \frac{\partial Q}{\partial y}\frac{\partial Y}{\partial v}\right)\frac{\partial Y}{\partial u} + Q\frac{\partial^2 Y}{\partial v \partial u} .$$

Subtracting, we obtain

$$\frac{\partial Q}{\partial x}\left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u}\right) = \frac{\partial Q}{\partial x} J(u,v) \quad ,$$

where $\partial Q/\partial x$ is evaluated at $F(u,v)$. Since $\partial Q/\partial x = f$, we have our desired result:

$$\iint_S f(x,y)\, dx\, dy = \pm \iint_T f(F(u,v))\, J(u,v)\, du\, dv. \quad \square$$

One can weaken the hypothesis of this theorem a bit if one wishes. Specifically, it is not necessary that the function $f(x,y)$ which is being integrated be continuous in an entire rectangle containing the region of integration S. It will suffice if $f(x,y)$ is merely continuous on some open set containing S and C. For it is a standard theorem (not too difficult to prove) that in this case one can find a function g that is continuous in the entire plane and equals f on S and C. One then applies the theorem to the function g.

$$\int_C Q\vec{j}\cdot d\underline{\alpha} = \int_a^b Q(\underline{\alpha}(t))\vec{j}\cdot\underline{\alpha}'(t)\ dt$$

$$= \int_a^b Q(\underline{\alpha}(t))\frac{d}{dt}Y(\underline{\beta}(t))\ dt$$

$$= \int_a^b Q(\underline{\alpha}(t))(\frac{\partial Y}{\partial u}\beta_1'(t) + \frac{\partial Y}{\partial v}\beta_2'(t))\ dt$$

$$= \int_a^b Q(F(\underline{\beta}(t)))\ [\frac{\partial Y}{\partial u}\vec{i} + \frac{\partial Y}{\partial v}\vec{j}]\cdot\underline{\beta}'(t)\ dt\ ,$$

where the partials are evaluated at $\underline{\beta}(t)$. We can write this last integral as a line integral over the curve D. Indeed, if we define

$$P_1(u,v) = Q(F(u,v))\frac{\partial Y}{\partial u}(u,v)\ ,$$

$$Q_1(u,v) = Q(F(u,v))\frac{\partial Y}{\partial v}(u,v)\ ,$$

then this last integral can be written as

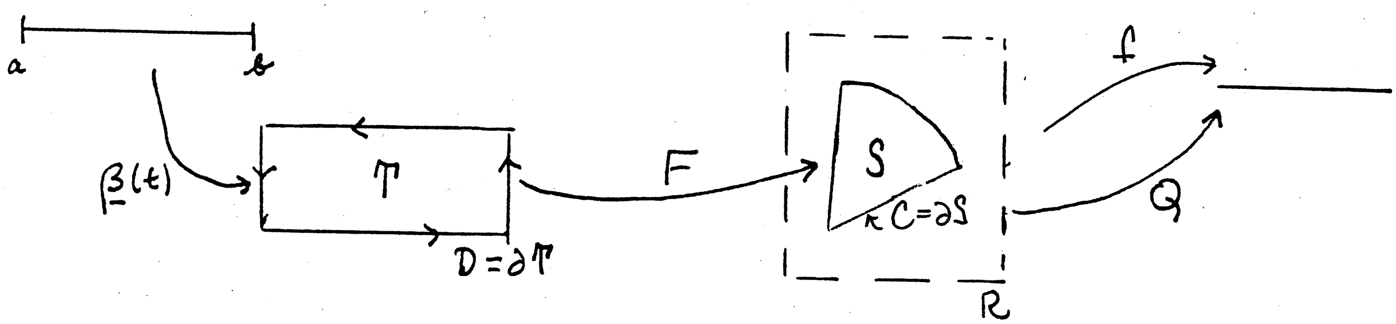
$$\int_D (P_1\vec{i}+Q_1\vec{j})\cdot d\underline{\beta}.$$

Now we apply Green's theorem to express this line integral as a double integral. Since T is by hypothesis a Green's region, this line integral equals

$$\iint_T (\frac{\partial Q_1}{\partial u} - \frac{\partial P_1}{\partial v})du\ dv.$$

It remains to compute these partials, using the chain rule. We have

Proof. Let $R = [c,d] \times [c',d']$. Define $Q(x,y) = \int_c^x f(t,y)dt$ for (x,y) in R. Then $\partial Q/\partial x = f(x,y)$ on all of R, because f is continuous. We prove our theorem by applying Green's theorem. Let $(u,v) = \underline{\beta}(t)$ be a parametrization of the curve D, for $a \le t \le b$; choose the counterclockwise direction, so Green's theorem holds for T. Then $\underline{\alpha}(t) = F(\underline{\beta}(t))$ is a parametrization of the curve C. It may be constant on some subintervals of the t-axis, but that doesn't matter when we compute line integrals. Also, it may be counterclockwise or clockwise.



We apply Green's theorem to S:

$$\iint_S f(x,y)\,dx\,dy = \iint_S \partial Q/\partial x\,dx\,dy = \pm \int_C (0\vec{i} + Q\vec{j})\cdot d\underline{\alpha}.$$

This sign is + if $\underline{\alpha}(t)$ parametrizes C in the counterclockwise direction, and − otherwise. Now let us compute this line integral.

The change of variables theorem

Theorem 7. (The change of variables theorem)

Let S be an open set in the (x,y) plane and let T be an open set in the (u,v) plane, bounded by the piecewise-differentiable simple closed curves C and D, respectively. Let $F(u,v) = (X(u,v), Y(u,v))$ be a (continuously differentiable) transformation from an open set of the (u,v) plane into the (x,y) plane that carries T into S, and carries $D = \partial T$ onto $C = \partial S$. As a transformation of D onto C, F may be constant on some segments of D, but otherwise is to be one-to-one.

Assume S and T are Green's regions. Assume that $f(x,y)$ is continuous in some rectangle R containing S. Then

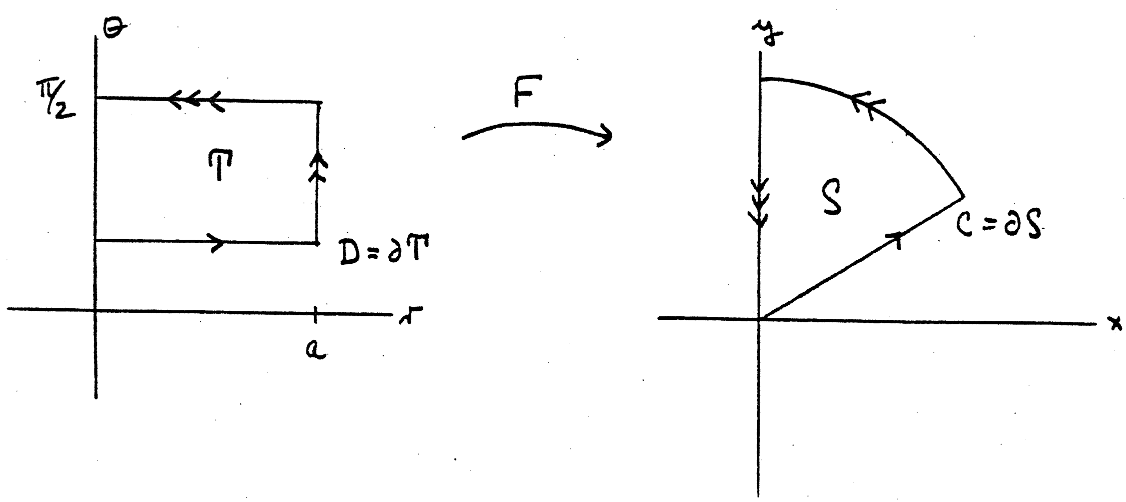
$$\iint_S f(x,y)\,dx\,dy = \pm \iint_T f(F(u,v))\,J(u,v)\,du\,dv\,.$$

Here $J(u,v) = \det \partial X,Y/\partial u,v$. The sign is $+$ if F carries the clockwise orientation of D to the clockwise orientation of C, and is $-$ otherwise.

Example 1. Consider the polar coordinate transformation

$$F(r,\theta) = (r\cos\theta,\ r\sin\theta)\,.$$

It carries the rectangle T in the (r,θ) plane indicated in the figure into the wedge S in the (x,y) plane. It is constant on the left edge of T, but is one-to-one on the rest of T. Note that it carries the counterclockwise orientation of $D = \partial T$ to the counterclockwise orientation of $C = \partial S$.



An alternate version of the change of variables theorem is the following:

Theorem 8. Assume all the hypotheses of the preceding theorem. Assume also that $J(u,v)$ does not change sign on the region T.

If $J(u,v) \geq 0$ on all of T, the sign in the change of variables formula is $+$; while if $J(u,v) \leq 0$ on all of T, the sign is $-$. Therefore in either case,

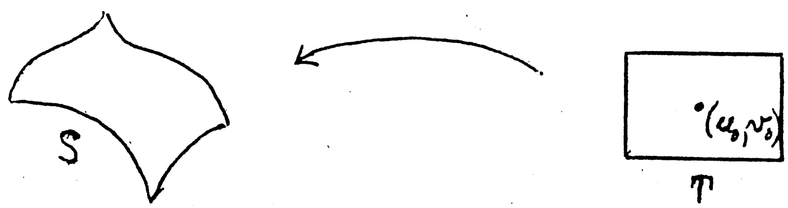
$$\iint_S f(x,y)\,dx\,dy = \iint_T f(F(u,v))\,|J(u,v)|\,du\,dv.$$

Proof. We apply the preceding theorem to the function $f(x,y) \equiv 1$. We obtain the formula

$$(*) \qquad \iint_S dx\,dy = \pm \iint_T J(u,v)\,du\,dv.$$

The left side of this equation is positive. Therefore if $J(u,v) \geq 0$ on all of T, the sign on the right side of the formula must be $+$; while if $J(u,v) \leq 0$ on all of T, the sign must be $-$. Now we recall that the sign does not depend on the particular function being integrated, only on the transformation involved. Then the theorem is proved. □

Remark. The formula we have just proved gives a geometric interpretation of the Jacobian determinant of a transformation. If $J(u,v) \neq 0$ at a particular point (u_0, v_0), let us choose a small rectangle T about this point, and consider its image S under the transformation. If T is small enough, $J(u,v)$ will



be very close to $J(u_0,v_0)$ on T, and so will not change sign. Assuming S is a Green's region, we have

$$\text{area } S = \iint_S dx\,dy = \iint_T |J(u,v)|\,du\,dv, \quad \text{so}$$

$$\text{area } S \sim |J(u_0,v_0)| \text{ (area } T).$$

Thus, roughly speaking, the magnitude of $J(u,v)$ measures how much the transformation stretches or shrinks areas as it carries a piece of the u, v plane to a piece of the x, y plane. And the sign of $J(u,v)$ tells whether the transformation preserves orientation or not; if the sign is negative, then the transformation "flips over" the region T before shrinking or stretching it to fit onto S.

Isometries

As an application of the change of variables theorem, we shall verify the final property of our notion of area, namely, the fact that congruent regions in the plane have the same area. First, we must make precise what we mean by a "congruence."

Definition. A transformation $\underline{h} : R^2 \longrightarrow R^2$ of the plane to itself is called a *congruence* or an *isometry* if it preserves distances between points. That is, $\underline{h}$ is a congruence if

$$\|\underline{h}(\underline{a}) - \underline{h}(\underline{b})\| = \|\underline{a}-\underline{b}\|$$

for every pair $\underline{a}$, $\underline{b}$ of points in the plane.

The following is a purely geometric result:

Lemma 9. *If* $\underline{h} : R^2 \longrightarrow R^2$ *is a congruence, then* h *has the form*

$$\underline{h}(x,y) = (ax + by + p, \ cx + dy + q)$$

or, writing vectors as column matrices,

$$\underline{h}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix},$$

where (a,c) *and* (b,d) *are unit orthogonal vectors. It follows that* $ad - bd$, *the Jacobian determinant of* $\underline{h}$, *equals* ± 1.

Proof. Let (p,q) denote the point $\underline{h}(0,0)$. Define $\underline{k} : R^2 \longrightarrow R^2$ by the equation

$$\underline{k}(x,y) = \underline{h}(x,y) - (p,q).$$

It is easy to check that $\underline{k}$ is a congruence, since

$$\underline{k}(\underline{a}) - \underline{k}(\underline{b}) = \underline{h}(\underline{a}) - \underline{h}(\underline{b})$$

for every pair of points $\underline{a}$, $\underline{b}$. Let us study the congruence $\underline{k}$, which has the property that $\underline{k}(\underline{0}) = \underline{0}$.

We first show that $\underline{k}$ preserves norms of vectors: By hypothesis,

$$\|\underline{a}-\underline{0}\| = \|\underline{k}(\underline{a}) - \underline{k}(\underline{0})\|, \quad \text{so}$$

$$\|\underline{a}\| = \|\underline{k}(\underline{a}) - \underline{0}\| = \|\underline{k}(\underline{a})\|.$$

Second, we show that $\underline{k}$ preserves dot products: By hypothesis,

$$\|\underline{k}(\underline{a}) - \underline{k}(\underline{b})\|^2 = \|\underline{a}-\underline{b}\|^2, \quad \text{so}$$

$$\|\underline{k}(\underline{a})\|^2 - 2\underline{k}(\underline{a})\cdot\underline{k}(\underline{b}) + \|\underline{k}(\underline{b})\|^2 = \|\underline{a}\|^2 - 2\underline{a}\cdot\underline{b} + \|\underline{b}\|^2.$$

Because $\underline{k}$ preserves norms, we must have

$$\underline{k}(\underline{a}) \cdot \underline{k}(\underline{b}) = \underline{a} \cdot \underline{b}.$$

We now show that $\underline{k}$ is a linear transformation. Let $\underline{e}_1$ and $\underline{e}_2$ be the usual unit basis vectors for R^2; then $(x,y) = x\underline{e}_1 + y\underline{e}_2$. Let

$$\underline{e}_3 = \underline{k}(\underline{e}_1) \quad \text{and} \quad \underline{e}_4 = \underline{k}(\underline{e}_2).$$

Then $\underline{e}_3$ and $\underline{e}_4$ are also unit orthogonal vectors, since $\underline{k}$ preserves dot products and norms. Given $\underline{x} = (x,y)$, consider the vector $\underline{k}(\underline{x})$; because $\underline{e}_3$ and $\underline{e}_4$ form a basis for R^2, we have

$$\underline{k}(\underline{x}) = \alpha(\underline{x})\underline{e}_3 + \beta(\underline{x})\underline{e}_4$$

for some scalars α and β, which are of course functions of $\underline{x}$. Let us compute α and β. We have

$$\begin{aligned}
\alpha(\underline{x}) &= \underline{k}(\underline{x}) \cdot \underline{e}_3 && \text{because } \underline{e}_3 \text{ is orthogonal to } \underline{e}_4, \\
&= \underline{k}(\underline{x}) \cdot \underline{k}(\underline{e}_1) && \text{by definition of } \underline{e}_3, \\
&= \underline{x} \cdot \underline{e}_1 && \text{because } \underline{k} \text{ preserves dot products,} \\
&= x && \text{because } \underline{e}_1 \text{ is orthogonal to } \underline{e}_2.
\end{aligned}$$

Similarly,

$$\beta(\underline{x}) = \underline{k}(\underline{x}) \cdot \underline{e}_4 = \underline{k}(\underline{x}) \cdot \underline{k}(\underline{e}_2) = \underline{x} \cdot \underline{e}_2 = y.$$

We conclude that for all points $\underline{x} = (x,y)$ of R^2,

$$\underline{k}(\underline{x}) = x\underline{e}_3 + y\underline{e}_4.$$

Letting $\underline{e}_3 = (a,c)$ and $\underline{e}_4 = (b,d)$, we can write $\underline{k}$ out in components in the form

$$\underline{k}(\underline{x}) = x(a,c) + y(b,d) = (ax + by,\ cx + dy).$$

Thus $\underline{k}$ is a linear transformation.

Returning now to our original transformation, $\underline{h}$, we recall that

$$\underline{k}(\underline{x}) = \underline{h}(\underline{x}) - (p,q).$$

Therefore we can write out $\underline{h}(\underline{x})$ in components as

$$\underline{h}(\underline{x}) = (ax + by + p,\ cx + dy + q).$$

To compute the Jacobian determinant of $\underline{h}$, we note that because $\underline{e}_3 = (a,c)$ and $\underline{e}_4 = (b,d)$ are unit orthogonal vectors, we have the equation

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{or}$$

$$(ad - bc)^2 = 1. \square$$

<u>Theorem 10</u>. <u>Let</u> h <u>be a congruence of the plane to itself, carrying region</u> S <u>to region</u> T. <u>If both</u> S <u>and</u> T <u>are Green's regions, then</u>

area S = area T.

Proof. The transformation carries the boundary of T in a one-to-one fashion onto the boundary of S (since distinct points of R^2 are carried by $\underline{h}$ to distinct points of R^2). Thus the hypotheses of the preceding theorem are satisfied. Furthermore, $|J(u,v)| = 1$. From the equation

$$\iint_S dx\, dy = \iint_T |J(u,v)|\, du\, dv$$

we conclude that

$$\text{area } S = \text{area } T. \quad \square$$

EXERCISES.

1. Let $\underline{h}(\underline{x}) = A \cdot \underline{x}$ be an arbitrary linear transformation of R^2 to itself. If S is a rectangle of area M, what is the area of the image of S under the transformation $\underline{h}$?

2. Given the transformation

$$\underline{h}(x,y) = (ax + by + p,\ cx + dy + q).$$

(a) Show that if (a,c) and (b,d) are unit orthogonal vectors, then $\underline{h}$ is a congruence.

(b) If $ad - bc = \pm 1$, show $\underline{h}$ preserves areas. Is $\underline{h}$ necessarily a congruence?

3. A *translation* of R^2 is a transformation of the form

$$\underline{g}(\underline{x}) = \underline{x} - \underline{p}$$

where $\underline{p}$ is fixed. A rotation of R^2 is a transformation of the form

$$\underline{h}(\underline{x}) = (x \cos \phi - y \sin \phi, \quad x \sin \phi + y \cos \phi),$$

where ϕ is fixed.

(a) Check that the transformation $\underline{h}$ carries the point with polar coordinates (r,θ) to the point with polar coordinates $(r,\theta+\phi)$.

(b) Show that translations and rotations are congruences. Conversely, show that every congruence with Jacobian $+1$ can be written as the composite of a translation and a rotation.

(c) Show that every congruence with Jacobian -1 can be written as the composite of a translation, a rotation, and the reflection map

$$\underline{k}(x,y) = (-x,y).$$

4. Let A be a square matrix. Show that if the rows of A are orthonormal vectors, then the columns of A are also orthonormal vectors.

5. Let S be the set of all (x,y) with $b^2x^2 + a^2y^2 \leq 1$. Given $f(x,y)$, express the integral $\iint_S f$ as an integral over the unit disc $u^2 + v^2 \leq 1$. Evaluate when $f(x,y) = x^2$.

6. Let C be a circular cylinder of radius a whose central axis is the x-axis. Let D be a circular cylinder of radius $b \leq a$ whose central axis is the z-axis. Express the volume common to the two cylinders as an integral in cylindrical coordinates. [Evaluate when $b = a$ – optional.]

7. Transform the integral in problem 3, p. D.26 by using the substitution $x = u/v$, $y = uv$ with $u, v > 0$. Evaluate the integral.

8. Let S be the parallelogram in the plane with vertices $(0,0)$ and $(1,3)$ and $(2,1)$ and $(3,4)$. Use a suitable linear transformation to transform the integral $\iint_S (x+2y)\,dx\,dy$ into an integral over the unit square $[0,1] \times [0,1]$. Evaluate.