Notes on double integrals.

(Read 11.1-11.5 of Apostol.)

Just as for the case of a single integral, we have the following condition for the existence of a double integral:

Theorem 1 (Riemann condition). Suppose f is defined on $Q = [a,b] \times [c,d]$. Then f is integrable on Q if and only if given any $\varepsilon > 0$, there are step functions s and t with $s \leqslant f \leqslant t$ on Q, such that

$$\iint_Q t - \iint_Q s < \varepsilon.$$

Let A be a number. If these step functions s and t satisfy the further condition that

$$\iint_Q s \leqslant A \leqslant \iint_Q t,$$

then $A = \iint_Q f.$

The proof is almost identical with the corresponding proof for the single integral.

Using this condition, one can readily prove the three basic properties--linearity, additivity, and comparison--for the integral $\iint_Q f$. We state them as follows:

Theorem 2. (a) Suppose f and g are integrable on Q. Then so is $cf(x) + dg(x)$; furthermore,

$$\iint_Q (cf + dg) = c \iint_Q f + d \iint_Q g.$$

(b) Let Q be subdivided into two rectangles Q_1 and Q_2. Then f is integrable over Q if and only if it is integrable over both Q_1 and Q_2; furthermore,

$$\iint_Q f = \iint_{Q_1} f + \iint_{Q_2} f.$$

(c) If $f \leq g$ on Q, and if f and g are integrable over Q, then

$$\iint_Q f \leq \iint_Q g.$$

To prove this theorem, one first verifies these results for step functions (see 11.3), and then uses the Riemann condition to prove them for general integrable functions. The proofs are very similar to those given for the single integral.

We give one of the proofs as an illustration. For example, consider the formula

$$\iint_Q (f + g) = \iint_Q f + \iint_Q g,$$

where f and g are integrable. We choose step functions s_1, s_2, t_1, t_2 such that

$$s_1 \leq f \leq t_1 \quad \text{and} \quad s_2 \leq g \leq t_2$$

on Q, and such that

$$\iint_Q (t_1 - s_1) < \varepsilon/2 \quad \text{and} \quad \iint_Q (t_2 - s_2) < \varepsilon/2.$$

We then find a single partition of Q relative to which all of s_1, s_2, t_1, t_2 are step functions; then $s_1 + s_2$ and $t_1 + t_2$ are also step functions relative to this partition. Furthermore, one adds the earlier inequalities to obtain

$$s_1 + s_2 \leqslant f + g \leqslant t_1 + t_2.$$

Finally, we compute

$$\iint_Q (t_1+t_2) - (s_1+s_2) = \iint_Q (t_1-s_1) + \iint_Q (t_2-s_2) < \varepsilon;$$

this computation uses the fact that linearity has already been proved for step functions. Thus $\iint_Q (f + g)$ exists. To calculate this integral, we note that

$$\iint_Q s_1 \leqslant \iint_Q f \leqslant \iint_Q t_1,$$

$$\iint_Q s_2 \leqslant \iint_Q g \leqslant \iint_Q t_2,$$

by definition. Then

$$\iint_Q (s_1+s_2) \leqslant \iint_Q f + \iint_Q g \leqslant \iint_Q (t_1+t_2);$$

here again we use the linearity of the double integral for step functions. It follows from the second half of the Riemann condition that $\iint_Q (f+g)$ must equal the number $A = \iint_Q f + \iint_Q g$.

Up to this point, the development of the double integral has been remarkably similar to the development of the single integral. Now things begin to change. We have the following basic questions to answer:

(1) Under what conditions does $\iint_Q f$ exist?

(2) If $\iint_Q f$ exists, how can one evaluate it?

(3) Is there a version of the substitution rule for double integrals?

(4) What are the applications of the double integral?

We shall deal with questions (1), (2), and (4) now, postponing question (3) until the next unit.

Let us tackle question (2) first. How can one evaluate the integral if one knows it exists? The answer is that such integrals can almost always be evaluated by repeated one-dimensional integration. More precisely, one has the following theorem:

Theorem 3 (Fubini theorem). *Let f be defined and bounded on a rectangle $Q = [a,b] \times [c,d]$, and assume that f is integrable on Q. For each fixed y in $[c,d]$, assume that the one-dimensional integral*

$$A(y) = \int_a^b f(x,y)\,dx$$

exists. Then the integral $\int_c^d A(y)\,dy$ exists, and furthermore,

$$\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \iint_Q f(x,y)\,dx\,dy.$$

<u>Proof</u>. We need to show that $\int_c^d A(y)dy$ exists and equals the double integral $\iint_Q f$.

Choose step functions $s(x,y)$ and $t(x,y)$, defined on Q, such that $s(x,y) \leqslant f(x,y) \leqslant t(x,y)$, and

$$\iint_Q t - \iint_Q s < \varepsilon.$$

This we can do because $\iint_Q f$ exists. For convenience, choose s and t so they are constant on the partition lines. (This does not affect their double integrals.) Then the one-dimensional integral

$$\int_a^b s(x,y)dx$$

exists. [For, given fixed y in $[c,d]$, the function $s(x,y)$ is either constant (if y is a partition point) or a step function of x; hence it is integrable.] Now I claim that the function $S(y) = \int_a^b s(x,y)dx$ is a step function on the interval $c \leq y \leq d$. For there are partitions $x_o,\dots,x_m$ and $y_o,\dots,y_n$ of $[a,b]$ and $[c,d]$, respectively, such that $s(x,y)$ is constant on each open rectangle $(x_{i-1},x_i) \times (y_{j-1},y_j)$. Let $\overline{y}$ and $\overline{\overline{y}}$ be any two points of the interval (y_{j-1},y_j). Then $s(x,\overline{y}) = s(x,\overline{\overline{y}})$ holds <u>for all</u> x. (This is immediate if x is in (x_{i-1},x_i); if x is a partition point, it follows from the fact that s is constant on the partition lines.) Therefore

$$\int_a^b s(x,\bar{y})\,dx = \int_a^b s(x,\bar{\bar{y}})\,dx.$$

Hence $S(y)$ is constant on (y_{j-1},y_j), so it is a step function.

A similar argument shows that the function

$$T(y) = \int_a^b t(x,y)\,dx$$

is a step function for $c \leq y \leq d$.

Now since $s \leq f \leq t$ for all (x,y), we have

$$\int_a^b s(x,y)\,dx \leq \int_a^b f(x,y)\,dx \leq \int_a^b t(x,y)\,dx,$$

by the comparison theorem. (The middle integral exists by hypothesis.) That is, for all y in $[c,d]$,

$$S(y) \leq A(y) \leq T(y).$$

Thus S and T are step functions lying beneath and above A, respectively. Furthermore

$$\iint_Q s = \int_c^d S(y)\,dy \qquad \text{and} \qquad \iint_Q t = \int_c^d T(y)\,dy,$$

(see p. 356), so that

$$\int_c^d T(y)\,dy - \int_c^d S(y)\,dy < \varepsilon.$$

It follows that $\int_c^d A(y)\,dy$ exists, by the Riemann condition.

Now that we know $A(y)$ is integrable, we can conclude from an earlier inequality that

$$\int_c^d S(y)\,dy \leq \int_c^d A(y)\,dy \leq \int_c^d T(y)\,dy;$$

that is,

$$\iint_Q s \leq \int_c^d A(y)\,dy \leq \iint_Q t.$$

But it is also true that

$$\iint_Q s \leq \iint_Q f \leq \iint_Q t,$$

by definition. Since the integrals of s and t are less than ε apart, we conclude that $\int_c^d A(y)\,dy$ and $\iint_Q f$ are within ε of each other. Because ε is arbitrary, they must be equal. □

With this theorem at hand, one can proceed to calculate some specific double integrals. Several examples are worked out in 11.7 and 11.8 of Apostol.

Now let us turn to the first of our basic questions, the one concerning the existence of the double integral. We readily prove the following:

<u>Theorem 4</u>. <u>The integral</u> $\iint_Q f$ <u>exists if</u> f <u>is continuous on the rectangle</u> Q.

<u>Proof</u>. All one needs is the small-span theorem of p. C.29.

Given ε', choose a partition of Q such that the span of f on each subrectangle of the partition is less than ε'. If Q_{ij} is a subrectangle, let

$$s_{ij} = \min f(x) \quad \text{on} \quad Q_{ij}; \qquad t_{ij} = \max f(x) \quad \text{on} \quad Q_{ij}.$$

Then $t_{ij} - s_{ij} < \varepsilon'$. Use the numbers s_{ij} and t_{ij} to obtain step functions s and t with $s \leq f \leq t$ on Q. One then has

$$\iint_Q (t - s) < \varepsilon'(d - c)(b - a).$$

This number equals ε if we begin the proof by setting $\varepsilon' = \varepsilon/(d-c)(b-a)$. □

In practice, this existence theorem is not nearly strong enough for our purposes, either theoretical or practical. We shall derive a theorem that is much stronger and more useful.

First, we need some definitions:

<u>Definition</u>. If $Q = [a,b] \times [c,d]$ is a rectangle, we define the **<u>area</u>** of Q by the equation

$$\text{area } Q = \iint_Q 1.$$

Of course, since 1 is a step function, we can calculate this integral directly as the product $(d-c)(b-a)$.

Additivity of $\iint$ implies that if we subdivide Q into two rectangles Q_1 and Q_2, then

$$\text{area } Q = \text{area } Q_1 + \text{area } Q_2.$$

Applying this formula repeatedly, we see that if one has a partition of Q, then

$$\text{area } Q = \textstyle\sum_{i,j} \text{area } Q_{ij},$$

where the summation extends over all subrectangles of the partition.

It now follows that if A and Q are rectangles and $A \subset Q$, then area $A \leq$ area Q.

<u>Definition</u>. Let D be a subset of the plane. Then D is said to have <u>content zero</u> if for every $\varepsilon > 0$, there is a finite set of rectangles whose union contains D and the sum of whose areas does not exceed ε.

<u>Examples</u>.

(1) A finite set has content zero.

(2) A horizontal line segment has content zero.

(3) A vertical line segment has content zero.

(4) A subset of a set of content zero has content zero.

(5) A finite union of sets of content zero has content zero.

(6) The graph of a continuous function

$$y = \phi(x); \quad a \leq x \leq b$$

has content zero.

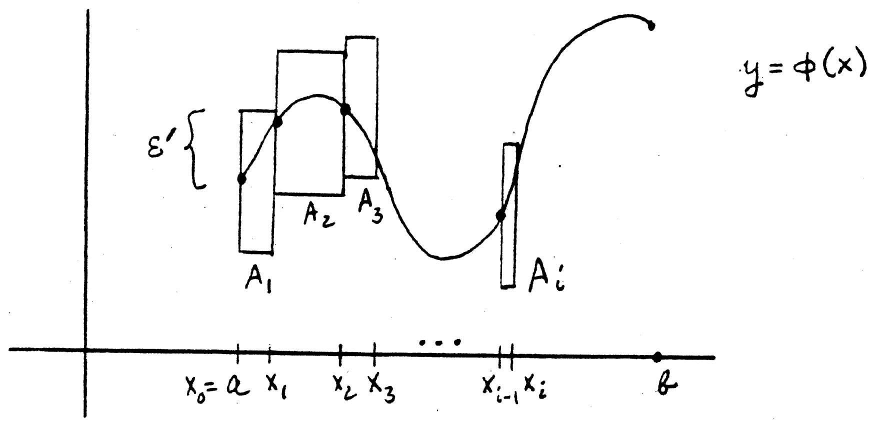
(7) The graph of a continuous function

$$x = \psi(y); \quad c \leqslant y \leqslant d$$

has content zero.

Most of these statements are trivial to prove; only the last two require some care. Let us prove (6). Let $\varepsilon' > 0$.

Given the continuous function ϕ, let us use the small-span theorem for functions of a single variable to choose a



partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$ such that the span of ϕ on each subinterval is less than ε'. Consider the rectangles

$$A_i = [x_{i-1}, x_i] \times [\phi(x_{i-1}) - \varepsilon', \phi(x_{i-1}) + \varepsilon']$$

for $i = 1,\ldots,n$. They cover the graph of ϕ, because $|\phi(x) - \phi(x_{i-1})| < \varepsilon'$ whenever x is in the interval $[x_{i-1}, x_i]$. The total area of the rectangles A_i equals

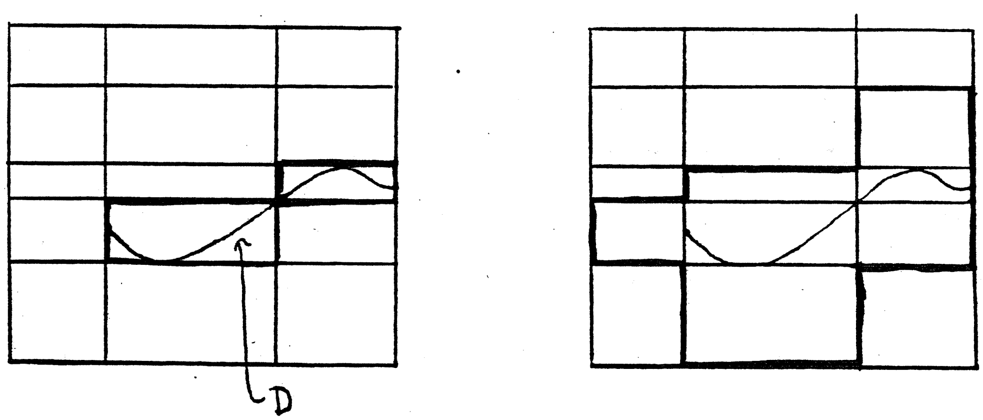
$$\sum_{i=1}^{n} (x_i - x_{i-1}) 2\varepsilon' = 2\varepsilon'(b - a).$$

This number equals ε if we begin the proof by setting $\varepsilon' = \varepsilon/2(b-a)$.

We now prove an elementary fact about sets of content zero:

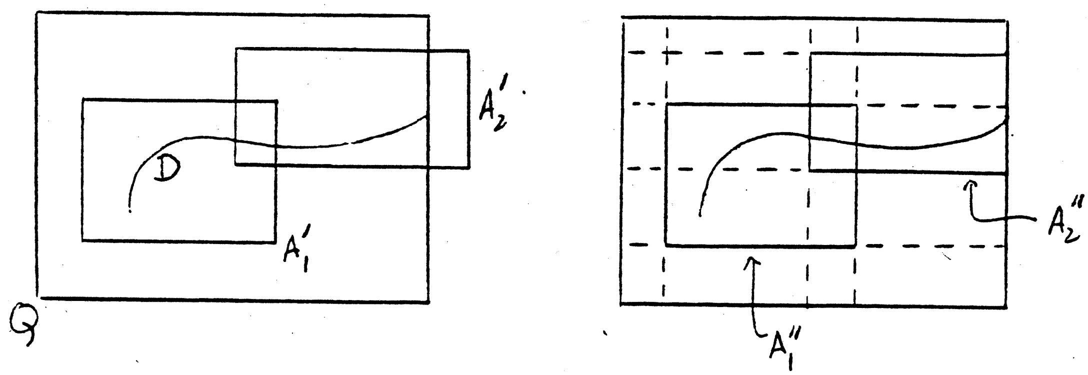
Lemma 5. *Let Q be a rectangle. Let D be a subset of Q that has content zero. Given $\varepsilon > 0$, there is a partition of Q such that those subrectangles of the partition that contain points of D have total area less than ε.*

Note that this lemma does not state merely that D is *contained in* the union of finitely many subrectangles of the partition having total area less than ε, but that the sum of the areas of *all* the subrectangles that contain points of D is less than ε. The following figure illustrates the distinction; D is contained in the union of two subrectangles, but there are seven subrectangles that contain points of D.



Proof. First, choose finitely many rectangles $A_1, \ldots, A_n$ of total area less than $\varepsilon/2$ whose union contains D. "Expand" each one slightly. That is, for each i, choose a rectangle A_i' whose interior contains A_i, such that the area of A_i' is no more than twice that of A_i. Then the union of the sets Int A_i' contains D, and the rectangles A_i' have total area less than ε. Of course, the rectangle A_i' may extend outside Q, so let A_i'' denote the rectangle that is the intersection of A_i' and Q. Then the rectangles A_i'' also have total area less than ε.

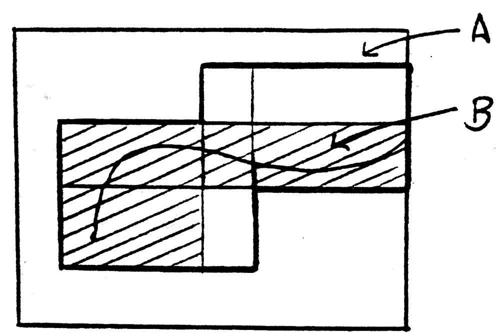
Now use the end points of the component intervals of the rectangles A_i'' to define a partition P of the rectangle Q. See the figure.



We show that this is our desired partition.

Note that by construction, the rectangle A_k'' is partitioned by P, so that it is a union of subrectangles Q_{ij} of P.

Now if a subrectangle Q_{ij} contains a point of D, then it contains a point of Int A_k' for some k, so that it actually lies in A_k' and hence in A_k''. Suppose we let B denote the union of all the subrectangles Q_{ij} that contain points of D; and let A be the union of the rectangles $A_1'',\ldots,A_n''$. Then $B \subset A$.



It follows that

$$\sum_{Q_{ij}\subset B} \text{area } Q_{ij} \leqslant \sum_{Q_{ij}\subset A} \text{area } Q_{ij}.$$

Now on the other hand, by additivity of area for rectangles,

$$\sum_{Q_{ij}\subset A_k''} Q_{ij} = \text{area } A_k''.$$

It follows that

$$\sum_{Q_{ij}\subset A} Q_{ij} \leqslant \sum_{k=1}^{n} \text{area } A_k''.$$

This last inequality is in general strict, because some subrectangles Q_{ij} belong to more than one rectangle A_k'', so their areas are counted more than once in the sum on the right side of the inequality.

It follows that

$$\sum_{Q_{ij}\subset B} \text{area } Q_{ij} < \varepsilon,$$

as desired. □

Now we prove our basic theorem on existence of the double integral $\iint_Q f$.

<u>Theorem 6</u>. <u>If</u> f <u>is bounded on</u> Q, <u>and is continuous on</u> Q <u>except on a set of content zero, then</u> $\iint_Q f$ <u>exists</u>.

<u>Proof</u>. <u>Step 1</u>. We prove a preliminary result:

Suppose that given $\epsilon > 0$, there exist functions g and h that are integrable over Q, such that

$$g(x) \leq f(x) \leq h(x) \qquad \text{for } x \text{ in } Q$$

and

$$\iint_Q h - \iint_Q g < \epsilon.$$

Then f is integrable over Q.

We prove this result as follows: Because h and g are integrable, we can find step functions s_1, s_2, t_1, t_2 such that

$$s_1 \leq g \leq t_1 \quad \text{and} \quad s_2 \leq h \leq t_2,$$

and such that

$$\iint_Q t_1 - \iint_Q s_1 < \epsilon \quad \text{and} \quad \iint_Q t_2 - \iint_Q s_2 < \epsilon\,.$$

Consider the step functions s_1 and t_2. We know that

$$s_1 \leq g \leq f \leq h \leq t_2$$

so s_1 is beneath f, and t_2 is above f. Furthermore, because the integral of g is between the integrals of s_1 and of t_1, we know that

$$\iint_Q g - \iint_Q s_1 < \epsilon.$$

Similarly,

$$\iint_Q t_2 - \iint_Q h < \epsilon.$$

If we add these inequalities and the inequality

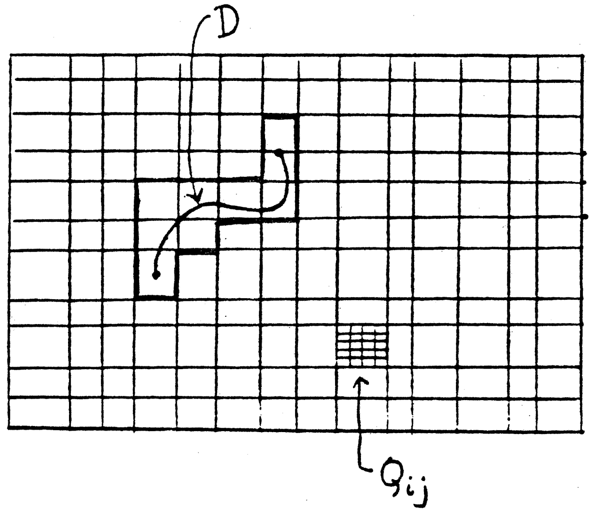
$$\iint_Q h - \iint_Q g < \epsilon,$$

we have

$$\iint_Q t_2 - \iint_Q s_1 < 3\epsilon.$$

Since ϵ is arbitrary, the Riemann condition is satisfied, so f is integrable over Q.

Step 2. Now we prove the theorem. Let D be a set of zero content containing the discontinuities of f. Choose M so that $|f(x)| \leq M$ for x in Q; then given $\epsilon > 0$, set $\epsilon' = \epsilon/2M$. Choose a partition P of Q such that those subrectangles that contain points of D have total area less than ϵ'. (Here we use the preceding lemma.)



Now we define functions g and h such that $g \leq f \leq h$ on Q. If Q_{ij} is one of the subrectangles that does not contain a point of D, set

$$g(x) = f(x) = h(x)$$

for $x \in Q_{ij}$. Do this for each such subrectangle. Then for any other x in Q, set

$$g(x) = -M \quad \text{and} \quad h(x) = M.$$

Then $g \leq f \leq h$ on Q.

Now g is integrable over each subrectangle Q_{ij} that does not contain a point of D, since it equals the continuous function f there. And g is integrable over each subrectangle Q_{ij} that does contain a point of D, because it is a step function on such a subrectangle. (It is constant on the interior of Q_{ij}.) The additivity property of the integral now implies that g is integrable over Q.

Similarly, h is integrable over Q. Using additivity, we compute the integral

$$\iint_Q (h-g) = \sum \iint_{Q_{ij}} (h-g) = 2M \sum (\text{area } Q_{ij} \text{ that contain points of D})$$

$$< 2M\epsilon' = \epsilon.$$

Thus the conditions of Step 1 hold, and f is integrable over Q. □

Theorem 7. Suppose f is bounded on Q, and f equals 0 except on a set D of content zero. Then $\iint_Q f$ exists and equals zero

Proof. We apply Step 2 of the preceding proof to the function f.

Choose M so that $|f(x)| \leq M$ for x in Q; given $\epsilon > 0$, set $\epsilon' = \epsilon/2M$. Choose a partition P such that those subrectangles that contain points of D have total area less than ϵ'.

Define functions g and h as follows: If Q_{ij} is one of the subrectangles that does not contain a point of D, set $g(x) = f(x) = 0$ and $h(x) = f(x) = 0$ on Q_{ij}. Do this for each such subrectangle. For any other x in Q, set

$$g(x) = -M \quad \text{and} \quad h(x) = M.$$

Then $g \leq f \leq h$ on Q.

Now g and h are step functions on Q, because they are constant on the interior of each subrectangle Q_{ij}. We compute

$$\iint_Q h = M\,(\textstyle\sum \text{(area } Q_{ij} \text{ that contain points of D))}$$
$$< 2M\epsilon' = \epsilon/2.$$

Similarly,

$$\iint_Q g > -M\epsilon' = -\epsilon/2.$$

Hence $\iint_Q (h-g) < \epsilon$, so that f is integrable over Q. Furthermore,

$$-\epsilon/2 < \iint_Q g \leq \iint_Q f \leq \iint_Q h < \epsilon/2.$$

Since ϵ is arbitrary, $\iint_Q f = 0$. □

Corollary 8. If $\iint_Q f$ exists, and if g is a bounded function that equals f except on a set of content zero, then $\iint_Q g$ exists and equals $\iint_Q f$.

Proof. We write $g = f + (g-f)$. Now f is integrable by hypothesis, and $g - f$ is integrable by the preceding corollary. Then g is integrable and

$$\iint_Q g = \iint_Q f + \iint_Q (g-f) = \iint_Q f. \square$$

Double integrals extended over more general regions.

(Read section 11.12 of Apostol.) In this section, Apostol defines $\iint_S f$ for a function f defined on a bounded set S, but then he quickly restricts himself to the special case where S is a region of Types I or II. We discuss here the general case.

First, we prove the following basic existence theorem:

Theorem 9. *Let S be a bounded set in the plane. If $\text{Bd } S$ has content zero, and if f is bounded on S and continuous at each point of $\text{Int } S$, then $\iint_S f$ exists.*

Proof. Let Q be a rectangle containing S. As usual, let $\tilde{f}$ equal f on S, and let $\tilde{f}$ equal 0 outside S. Then $\tilde{f}$ is continuous at each point x_0 of the interior of S (because it equals f in an open ball about x_0, and f is continuous at x_0). The function $\tilde{f}$ is also continuous at each point x_1 of the exterior of S, because it equals zero on an open ball about x_1. The only points where $\tilde{f}$ can fail to be continuous are points of the boundary of S, and this set, by assumption, has content zero. Hence $\iint_Q \tilde{f}$ exists. $\square$

Note: Adjoining or deleting boundary points of S changes the value of f only on a set of content zero, so that value of $\iint_S f$ remains unchanged. Thus $\iint_S f = \iint_{\text{Int } S} f$, for instance.

Let us remark on a more general existence theorem than that stated in Theorem 9. If S is a bounded set, and if $\text{Bd } S$ has content zero, and if f is continuous on $\text{Int } S$ except on a set D of content zero, then $\iint_S f$ exists. For in this case the discontinuities of the extended function $\tilde{f}$ lie in the union of the sets $\text{Bd } S$ and D, and this set has content zero because both $\text{Bd } S$ and D do.

There are more general existence theorems even than this, but we shall not consider them.

Now we note that the basic properties of the double integral hold also for this extended integral:

Theorem 10. *Let S be a bounded set in the plane. One has the following properties:*

(a) *Linearity.*

$$\iint_S cf + dg = c \iint_S f + d \iint_S g;$$

the left side exists if the right side does.

(b) *Comparison.* If $f \leq g$ on the set S, then

$$\iint_S f \leq \iint_S g,$$

provided both integrals exist.

(c) Additivity. Let $S = S_1 \cup S_2$. If $S_1 \cap S_2$ has content zero, then

$$\iint_S f = \iint_{S_1} f + \iint_{S_2} f,$$

provided the right side exists.

Proof. (a) Given f, g defined on S, let $\tilde{f}, \tilde{g}$ equal f, g, respectively, on S and equal 0 otherwise. Then $c\tilde{f} + d\tilde{g}$ equals $cf + dg$ on S and 0 otherwise. Let Q be a rectangle containing S. We know that

$$\iint_Q c\tilde{f} + d\tilde{g} = c \iint_Q \tilde{f} + d \iint_Q \tilde{g};$$

from this linearity follows.

(b) Similarly, if $f \leq g$, then $\tilde{f} \leq \tilde{g}$, from which we conclude that

$$\iint_S f = \iint_Q \tilde{f} \leq \iint_Q \tilde{g} = \iint_S g.$$

(c) Let Q be a rectangle containing S. Let f_1 equal f on S_1, and equal 0 elsewhere. Let f_2 equal f on S_2, and equal 0 elsewhere. Let f_3 equal f on S, and equal 0 elsewhere. Consider the function

$$f_4 = f_1 + f_2 - f_3;$$

it equals f on the set $S_1 \cap S_2$, and equals zero elsewhere. Because $S_1 \cap S_2$ has content zero, $\iint_Q f_4$ exists and equals zero. Now

$$f_3 = f_1 + f_2 - f_4;$$

linearity implies that

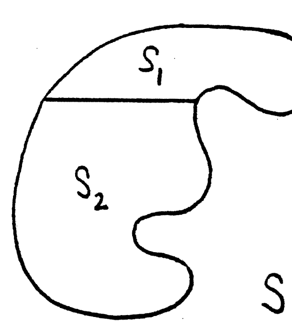
$$\iint_Q f_3 = \iint_Q f_1 + \iint_Q f_2 - \iint_Q f_4,$$

or

$$\iint_S f = \iint_{S_1} f + \iint_{S_2} f. \quad \square$$

How can one evaluate $\iint_S f$ when S is a general region? The computation is easy when S is a region of type I or II and f is continuous on the interior of S; one evaluates $\iint_S f$ by iterated integration. This result is proved on p. 367 of Apostol.

Using additivity, one can also evaluate $\iint_S f$ for many other regions as well. For example, to integrate a continuous function f over the region S pictured, one can



break it up as indicated into two regions S_1 and S_2 that intersect in a set of content zero. Since S_1 is of type I and S_2 is of type II, we can compute the integrals $\iint_{S_1} f$ and $\iint_{S_2} f$ by iterated integration. We add the results to obtain $\iint_S f$.

Area.

We can now construct a rigorous theory of area. We already have defined the area of the rectangle $Q = [a,b] \times [c,d]$ by the equation

$$\text{area } Q = \iint_Q 1.$$

We use this same equation for the general definition.

Definition. Let S be a bounded set in the plane. We say that S is Jordan-measurable if $\iint_S 1$ exists; in this case, we define

$$\text{area } S = \iint_S 1.$$

Note that if $\text{Bd } S$ has content zero, then S is Jordan-measurable, by by Theorem 9. The converse also holds; the proof is left as an exercise.

The area function has the following properties:

Theorem 11. Let S and T be Jordan-measurable sets in the plane.

(1) (Monotonicity). If $S \subset T$, then area $S \leq$ area T.

(2) (Positivity). Area $S \geq 0$, and equality holds if and only if S has content zero.

(3) (Additivity) If $S \cap T$ is a set of content zero, then $S \cup T$ is Jordan-measurable and

$$\text{area}(S \cup T) = \text{area } S + \text{area } T.$$

(4) Area S = Area(Int S) = Area($S \cup \text{Bd } S$).

Proof. Let Q be a rectangle containing S and T. Let

$$\begin{aligned} 1_S(x) &= 1 \quad \text{for} \quad x \in S \\ &= 0 \quad \text{for} \quad x \notin S. \end{aligned}$$

Define 1_T similarly.

(1) If S is contained in T, then $1_S(x) \leq 1_T(x)$. Then by the comparison theorem,

$$\text{area } S = \iint_S 1 = \iint_Q 1_S \leq \iint_Q 1_T = \iint_T 1 = \text{area } T.$$

(2) Since $0 < 1$, we have by the comparison theorem,

$$0 = \iint_S 0 \leq \iint_S 1 = \text{area } S,$$

for all S. If S has content zero, then $\iint_S 1 = \iint_Q 1_S = 0$, by Corollary 7.

Conversely, suppose $\iint_S 1 = 0$. Then $\iint_Q 1_S = 0$. Given $\varepsilon > 0$, there must be a step function $t \geqslant 1_S$ defined on Q such that $\iint_Q t < \varepsilon$. Let P be a partition relative to which t is a step function. Now if a subrectangle Q_{ij} of this partition contains a point of S in its interior, then the value of t on this subrectangle must be at least 1. Thus these subrectangles have total area less than ε. Now S is contained in the union of these subrectangles (of total area less than ε) and the partition lines. Thus S has content zero.

(3) Because $\iint_S 1$ and $\iint_T 1$ exist and $S \cap T$ has content zero, it follows from additivity that $\iint_{S \cup T} 1$ exists and equals $\iint_S 1 + \iint_T 1$.

(4) Since the part of S not in $\text{Int } S$ lies in $\text{Bd } S$, it has content zero. Then additivity implies that

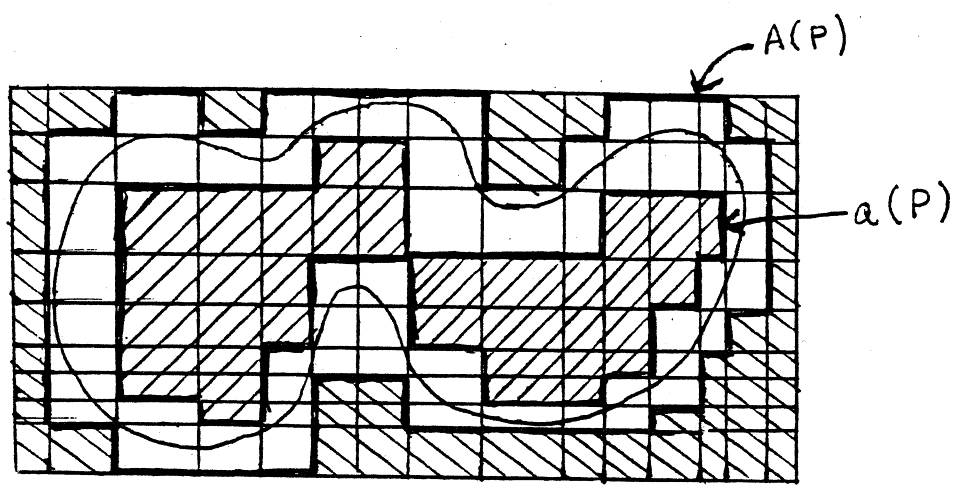
$$\begin{aligned}\text{area } S &= \text{area}(\text{Int } S) + \text{area}(S - \text{Int } S)\\ &= \text{area}(\text{Int } S).\end{aligned}$$

A similar remark shows that

$$\begin{aligned}\text{area}(S \cup \text{Bd } S) &= \text{area}(\text{Int } S) + \text{area}(\text{Bd } S)\\ &= \text{area}(\text{Int } S). \quad \square\end{aligned}$$

Remark. Let S be a bounded set in the plane. A direct way of defining the area of S, without developing integration theory, is as follows: Let Q be a rectangle containing S.

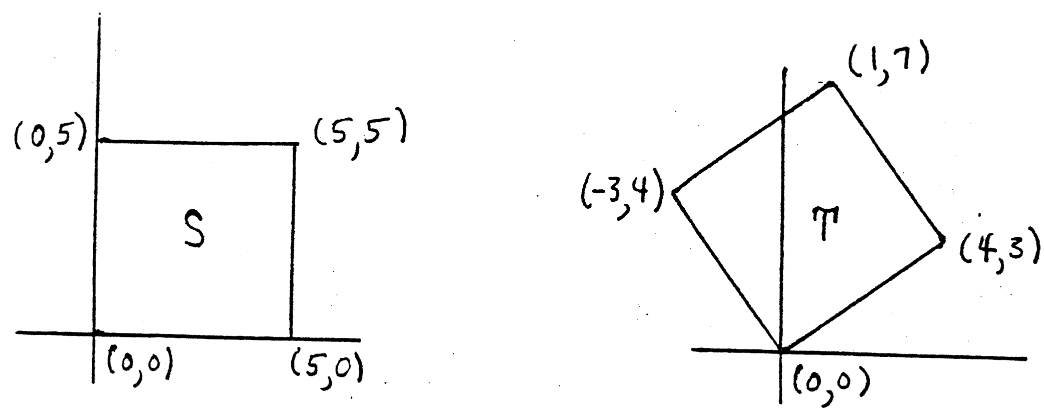
Given a partition P of Q, let $a(P)$ denote the total area of all subrectangles of P that are contained in S, and let $A(P)$ denote the total area of all subrectangles of P that contain points of S. Define the inner area of S be the supremum



of the numbers $a(P)$, as P ranges over all partitions of Q; and define the outer area of S to be the infemum of the numbers $A(P)$. If the inner area and outer area of S are equal, their common value is called the area of S.

We leave it as a (not too difficult) exercise to show that this definition of area is the same as the one we have given.

Remark. There is just one fact that remains to be proved about our notion of area. We would certainly wish it to be true that if two sets S and T in the plane are "congruent" in the sense of elementary geometry, then their areas are the same. This fact is not immediate from the definition of area, for we used rectangles with sides parallel to the coordinate axes to form the partitions on which we based our notion of "integral", and hence of "area". It is not immediate, for instance, that the rectangles S and T pictured below have the same area, for the area of T is defined



by approximating T by rectangles with vertical and horizontal sides. [Of course, we can write equations for the curves bounding T and compute its area by integration, if we wish.]

Proof of the invariance of area under "congruence" will have to wait until we study the problem of change of variables in a double integral.

Exercises

1. Show that if $\iint_S 1$ exists, then Bd S has content zero. [Hint: Choose Q so that $S \subset Q$. Since $\iint_Q 1_S$ exists, there are functions s and t that are step functions relative to a partition P of Q, such that $s \leq 1_S \leq t$ on Q and $\iint_Q (t - s) < \varepsilon$. Show that the subrectangles determined by P that contain points of S have total volume less than ε.]

2. (a) Let S and T be bounded subsets of R^2. Show that Bd $(S \cup T) \subset$ (Bd $S \cup$ Bd T). Give an example where equality does not hold.

(b) Show that if S and T are Jordan-measurable, then so are $S \cup T$ and $S \cap T$, and furthermore

$$\text{area}(S \cup T) = \text{area } S + \text{area } T - \text{area } (S \cap T).$$

3. Express in terms of iterated integrals the double integral $\iint_S x^2y^2$, where S is the bounded portion of the first quadrant lying between the curves $xy = 1$ and $xy = 2$ and the lines $y = x$ and $y = 4x$. (Do not evaluate the integrals.)

4. A solid is bounded above by the surface $z = x^2 - y^2$, below by the xy-plane, and by the plane $x = 2$. Make a sketch; express its volume as an integral; and find the volume.

5. Express in terms of iterated integrals the volume of the region in the first octant of R^3 bounded by: (a) The surfaces $z = xy$ and $z = 0$ and $x + 2y + z = 1$. (b) The surfaces $z = xy$ and $z = 0$ and $x + 2y - z = 1$.

Let Q denote the rectangle $[0,1] \times [0,1]$ in the following exercises.

6. (a) Let $f(x,y) = 1/(y-x)$ if $x \neq y$,

$f(x,y) = 0$ if $x = y$.

Does $\iint_Q f$ exist?

(b) Let $g(x,y) = \sin(1/(y-x))$ if $x \neq y$,

$g(x,y) = 0$ if $x = y$.

Does $\iint_Q g$ exist?

7. Let $f(x,y) = 1$ if $x = 1/2$ and y is rational,

$f(x,y) = 0$ otherwise

Show that $\iint_Q f$ exists but $\int_0^1 f(x,y)dy$ fails to exist when $x = 1/2$.

8. Let $f(x,y) = 1$ if (x,y) has the form $(a/p,b/p)$,

where a and b are integers and p is prime,

$f(x,y) = 0$ otherwise.

Show that $\int_0^1 \int_0^1 f(x,y)dy\,dx$ exists but $\iint_Q f$ does not.

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