## The inverse of a matrix

We now consider the problem of the existence of multiplicatiave inverses for matrices. At this point, we must take the non-commutativity of matrix multiplication into account. For it is perfectly possible, given a matrix $A$, that there exists a matrix $B$ such that $A \cdot B$ equals an identity matrix, without it following that $B \cdot A$ equals an identity matrix. Consider the following example:

Example 6. Let $A$ and $B$ be the matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then $A \cdot B = I_2$, but $B \cdot A \neq I_3$, as you can check.

Definition. Let $A$ be a $k$ by $n$ matrix. A matrix $B$ of size $n$ by $k$ is called an inverse for $A$ if both of the following equations hold:

$$A \cdot B = I_k \qquad \text{and} \qquad B \cdot A = I_n .$$

We shall prove that if $k \neq n$, then it is impossible for both these equations to hold. Thus only square matrices can have inverses. We also show that if the matrices are square and one of these equations holds, then the other equation holds as well!

Theorem 13. Let $A$ be a matrix of size $k$ by $n$. Then $A$ has an inverse if and only if $k = n = \text{rank } A$. If $A$ has an inverse, that inverse is unique.

Proof. Step 1. If $B$ is an $n$ by $k$ matrix, we say $B$ is a right inverse for $A$ if $A \cdot B = I_k$. We say $B$ is a left inverse for $A$ if $B \cdot A = I_n$. Let $A$ be a $k$ by $n$ matrix.

Let $r$ be the rank of $A$. We show that if $A$ has a right inverse, then $r = k$; and if $A$ has a left inverse, then $r = n$. The "only if" part of the theorem follows.

First, suppose $B$ is a right inverse for $A$. Then $A \cdot B = I_k$. It follows that the system of equations $A \cdot X = C$ has a solution for arbitrary $C$, for the vector $X = B \cdot C$ is one such solution, as you can check. Theorem 6 then implies that $r$ must equal $k$.

Second, suppose $B$ is a left inverse for $A$. Then $B \cdot A = I_n$. It follows that the system of equations $A \cdot X = \underline{0}$ has only the trivial solution, for the equation $A \cdot X = \underline{0}$ implies that $B \cdot (A \cdot X) = \underline{0}$, whence $X = \underline{0}$. Now the dimension of the solution space of the system $A \cdot X = \underline{0}$ is $n - r$; it follows that $n - r = 0$.

Step 2. Now let $A$ be an $n$ by $n$ matrix of rank $n$. We show there is a matrix $B$ such that $A \cdot B = I_n$.

Because the rows of $A$ are independent, the system of equations $A \cdot X = C$ has a solution for arbitrary $C$. In particular, it has a solution when $C$ is one of the unit coordinate vectors $E_i$ in $V_n$. Let us choose $B_i$ so that

$$A \cdot B_i = E_i ,$$

for $i = 1,\ldots,n$. Then if $B$ is the $n$ by $n$ matrix whose successive columns are $B_1,\ldots,B_n$, the product $A \cdot B$ equals the matrix whose successive columns are $E_1,\ldots,E_n$; that is, $A \cdot B = I_n$.

Step 3. We show that if $A$ and $B$ are $n$ by $n$ matrices and $A \cdot B = I_n$, then $B \cdot A = I_n$. The "if" part of the theorem follows.

Let us note that if we apply Step 1 to the case of a square matrix of size $n$ by $n$, it says that if such a matrix has either a right inverse <u>or</u> a left inverse, then its rank must be $n$.

Now the equation $A \cdot B = I_n$ says that $A$ has a right inverse and that $B$ has a left inverse. Hence both $A$ and $B$ must have rank $n$. Applying Step 2 to the matrix $B$, we see that there is a matrix $C$ such that $B \cdot C = I_n$. Now we compute

$$\begin{aligned} A\cdot(B\cdot C) &= (A\cdot B)\cdot C , \\ A\cdot I_n &= I_n\cdot C , \\ A &= C . \end{aligned}$$

The equation $B \cdot C = I_n$ now becomes $B \cdot A = I_n$, as desired.

<u>Step</u> 4. The computation we just made shows that if a matrix has an inverse, that inverse is unique. Indeed, we just showed that if $B$ has an left inverse $A$ and a right inverse $C$, then $A = C$. □

Let us state the result proved in Step 3 as a separate theorem:

<u>Theorem</u> 14. If $A$ and $B$ are $n$ by $n$ matrices such that $A \cdot B = I_n$, then $B \cdot A = I_n$. □

We now have a theoretical criterion for the existence of $A^{-1}$. But how can one find $A^{-1}$ in practice? For instance, how does one compute $B = A^{-1}$ if $A$ is a given nonsingular 3 by 3 matrix? By Theorem 14, it will suffice to find a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

such that $A \cdot B = I_3$. But this problem is just the problem of solving three systems of linear equations

$$A \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Thus the Gauss-Jordan algorithm applies. An efficient way to apply this algorithm to the computation of $A^{-1}$ is outlined on p. 612 of Apostol, which you should read now.

There is also a formula for $A^{-1}$ that involves determinants. It is given in the next section.

Remark . It remains to consider the question whether the existence of the inverse of a matrix has any practical significance, or whether it is of theoretical interest only. In fact, the problem of finding the inverse of a matrix in an efficient and accurate way is of great importance in engineering. One way to explain this is to note that often in a real-life situation, one has a fixed matrix $A$, and one wishes to solve the system $A \cdot X = C$ repeatedly, for many different values of $C$. Rather than solving each one of these systems separately, it is much more efficient to find the inverse of $A$, for then the solution $X = A^{-1} \cdot C$ can be computed by simple matrix multiplication.

Exercises

1. Give conditions on $a,b,c,d,e,f$ such that the matrix

$$B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

is a right inverse to the matrix $A$ of Example 6. Find two right inverses for $A$.

2. Let $A$ be a $k$ by $n$ matrix with $k < n$. Show that $A$ has no left inverse. Show that if $A$ has a right inverse, then that right inverse is not unique.

3. Let $B$ be an $n$ by $k$ matrix with $k < n$. Show that $B$ has no right inverse. Show that if $B$ has a left inverse, then that left inverse is not unique.

## Determinants

The determinant is a function that assigns, to each square matrix $A$, a real number. It has certain properties that are expressed in the following theorem:

Theorem 15. There exists a function that assigns, to each $n$ by $n$ matrix $A$, a real number that we denote by $\det A$. It has the following properties:

(1) If $B$ is the matrix obtained from $A$ by exchanging rows $i$ and $j$ of $A$, then $\det B = -\det A$.

(2) If $B$ is the matrix obtained form $A$ by replacing row $i$ of $A$ by itself plus a scalar multiple of row $j$ (where $i \neq j$), then $\det B = \det A$.

(3) If $B$ is the matrix obtained from $A$ by multiplying row $i$ of $A$ by the scalar $c$, then $\det B = c \cdot \det A$.

(4) If $I_n$ is the identity matrix, then $\det I_n = 1$.

We are going to assume this theorem for the time being, and explore some of its consequences. We will show, among other things, that these four properties characterize the determinant function completely. Later we shall construct a function satisfying these properties.

First we shall explore some consequences of the first three of these properties. We shall call properties (1)-(3) listed in Theorem 15 the elementary row properties of the determinant function.

Theorem 16. Let $f$ be a function that assigns, to each $n$ by $n$ matrix $A$, a real number. Suppose $f$ satisfies the elementary row properties of the determinant function. Then for every $n$ by $n$ matrix $A$,

$$(*) \qquad f(A) = f(I_n) \cdot \det A .$$

This theorem says that any function $f$ that satisfies properties (1), (2), and (3) of Theorem 15 is a scalar multiple of the determinant function. It also says that if $f$ satisfies property (4) as well, then $f$ must equal the determinant function. Said differently, there is <u>at most one</u> function that satisfies all four conditions.

<u>Proof</u>. <u>Step</u> 1. First we show that if the rows of $A$ are dependent, then $f(A) = 0$ and $\det A = 0$. Equation (*) then holds trivially in this case.

Let us apply elementary row operations to $A$ to bring it to echelon form $B$. We need only the first two elementary row operations to do this, and they change the values of $f$ and of the determinant function by at most a sign. Therefore it suffices to prove that $f(B) = 0$ and $\det B = 0$. The last row of $B$ is the zero row, since $A$ has rank less than $n$. If we multiply this row by the scalar $c$, we leave the matrix unchanged, and hence we leave the values of $f$ and $\det$ unchanged. On the other hand, this operation multiplies these values by $c$. Since $c$ is arbitrary, we conclude that $f(B) = 0$ and $\det B = 0$.

<u>Step</u> 2. Now let us consider the case where the rows of $A$ are independent. Again, we apply elementary row operations to $A$. However, we will do it very carefully, so that the values of $f$ and $\det$ do not change.

As usual, we begin with the first column. If all entries are zero, nothing remains to be done with this column. We move on to consider columns $2,\ldots,n$ and begin the process again.

Otherwise, we find a non-zero entry in the first column. If necessary, we exchange rows to bring this entry up to the upper left-hand corner; this changes the sign of both the functions $f$ and $\det$, so we then multiply this row by $-1$ to change the signs back. Then we add multiples of the first row to each of the remaining rows so as to make all the remaining entries in the first column into zeros. By the preceding theorem and its corollary, this does not change the values of either $f$ or $\det$.

Then we repeat the process, working with the second column and with rows $2,\ldots,n$. The operations we apply will not affect the zeros we already have in column 1.

Since the rows of the original matrix were independent, then we do not have a zero row at the bottom when we finish, and the "stairsteps" of the echelon form go over just one step at a time.

In this case, we have brought the matrix to a form where all of the entries below the main diagonal are zero. (This is what is called <u>upper triangular form</u>.) Furthermore, all the diagonal entries are non-zero. Since the values of $f$ and $\det$ remain the same if we replace $A$ by this new matrix $B$, it now suffices to prove our formula for a matrix of the form

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}.$$

where the diagonal entries are non-zero.

Step 3. We show that our formula holds for the matrix B. To do this we continue the Gauss-Jordan elimination process. By adding a multiple of the last row to the rows above it, then adding multiples of the next-to-last row to the rows lying above it, and so on, we can bring the matrix to the form where all the non-diagonal entries vanish. This form is called diagonal form. The values of both $f$ and det remain the same if we replace B by this new matrix C. So now it suffices to prove our formula for a matrix of the form

$$C = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix}.$$

(Note that the diagonal entries of B remain unchanged when we apply the Gauss-Jordan process to eliminate all the non-zero entries above the diagonal. Thus the diagonal entries of C are the same as those of B.)

We multiply the first row of C by $1/b_{11}$. This action multiplies the values of both $f$ and det by a factor of $1/b_{11}$. Then we multiply the second row by $1/b_{22}$, the third by $1/b_{33}$, and so on. By this process, we transform the matrix C into the identity matrix $I_n$. We conclude that

$$f(I_n) = (1/b_{11})\dots(1/b_{nn})\, f(C)\ , \quad \text{and}$$

$$\det I_n = (1/b_{11})\dots(1/b_{nn}) \det C.$$

Since $\det I_n = 1$ by hypothesis, it follows from the second equation that

$$\det C = b_{11}\, b_{22} \cdots b_{nn}\ .$$

Then it follows from the first equation that

$$f(C) = f(I_n) \cdot \det C,$$

as desired. $\square$

Besides proving the determinant function unique, this theorem also tells us one way to compute determinants. One applies this version of the Gauss-Jordan algorithm to reduce the matrix to echelon form. If the matrix that results has a zero row, then the determinant is zero. Otherwise, the matrix that results is in upper triangular form with non-zero diagonal entries, and the determinant is the product of the diagonal entries.

The proof of this theorem tells us something else: If the rows of $A$ are not independent, then $\det A = 0$, while if they are independent, then $\det A \neq 0$. We state this result as a theorem:

<u>Theorem</u> 16. Let $A$ be an $n$ by $n$ matrix. Then $A$ has rank $n$ if and only if $\det A \neq 0$. $\square$

An $n$ by $n$ matrix $A$ for which $\det A \neq 0$ is said to be <u>non-singular</u>. This theorem tells us that $A$ has rank $n$ if and only if $A$ is non-singular.

Now we prove a totally unexpected result:

<u>Theorem</u> 17. Let $A$ and $B$ be $n$ by $n$ matrices. Then

$$\det(A \cdot B) = (\det A) \cdot (\det B).$$

<u>Proof.</u> This theorem is almost impossible to prove by direct computation. Try the case $n = 2$ if you doubt me ! Instead, we proceed in another direction:

Let $B$ be a fixed $n$ by $n$ matrix. Let us define a function $f$ of $n$ by $n$ matrices by the formula

$$f(A) = \det(A \cdot B).$$

We shall prove that $f$ satisfies the elementary row properties of the determinant function. From this it follows that

$$f(A) = f(I_n)\cdot \det A ,$$

which means that

$$\begin{aligned}\det(A\cdot B) &= \det(I_n\cdot B)\cdot \det A\\ &= \det B \cdot \det A ,\end{aligned}$$

and the theorem is proved.

First, let us note that if $A_1,\ldots,A_n$ are the rows of $A$, considered as row matrices, then the rows of $A\cdot B$ are (by the definition of matrix multiplication) the row matrices $A_1\cdot B,\ldots,A_n\cdot B$. Now exchanging rows $i$ and $j$ of $A$, namely $A_i$ and $A_j$, has the effect of exchanging rows $i$ and $j$ of $A\cdot B$. Thus this operation changes the value of $f$ by a factor of $-1$. Similarly, replacing the $i^{th}$ row $A_i$ of $A$ by $A_i + cA_j$ has the effect on $A\cdot B$ of replacing its $i^{th}$ row $A_i\cdot B$ by

$$\begin{aligned}(A_i + cA_j)\cdot B &= A_i\cdot B + c\, A_j\cdot B\\ &= (\text{row } i \text{ of } A\cdot B) + c(\text{row } j \text{ of } A\cdot B).\end{aligned}$$

Hence it leaves the value of $f$ unchanged. Finally, replacing the $i^{th}$ row $A_i$ of $A$ by $cA_i$ has the effect on $A\cdot B$ of replacing the $i^{th}$ row $A_i\cdot B$ by

$$(cA_i)\cdot B = c\,(A_i\cdot B) = c\,(\text{row } i \text{ of } A\cdot B).$$

Hence it multiplies the value of $f$ by $c$. $\square$

The determinant function has many further properties, which we shall not explore here. (One reference book on determinants runs to four volumes!) We shall derive just one additional result, concerning the inverse matrix.

Exercises

1. Suppose that $f$ satisfies the elementary row properties of the determinant function. Suppose also that $x, y, z$ are numbers such that

$$f\begin{bmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = 1,$$

Compute the value of $f$ for each of the following matrices:

(a) $\begin{bmatrix} 2x & 2y & 2z \\ 3/2 & 0 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+2 & y+2 & z+2 \end{bmatrix}$ (c) $\begin{bmatrix} x-1 & y-1 & z-1 \\ 1 & 1 & 1 \\ 4 & 1 & 3 \end{bmatrix}$

2. Let $f$ be the function of Exercise 1. Calculate $f(I_n)$. Express $f$ in terms of the determinant function.

3. Compute the determinant of the following matrix, using Gauss-Jordan elimination.

$$\begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

4. Determine whether the following sets of vectors are linearly independent, using determinants.

(a) $A_1 = (1,-1,0)$, $A_2 = (0,1,-1)$, $A_3 = (2,3,-1)$.

(b) $A_1 = (1,-1,2,1)$, $A_2 = (-1,2,-1,0)$, $A_3 = (3,-1,1,0)$, $A_4 = (1,0,0,1)$.

(c) $A_1 = (1,0,0,0,1)$, $A_2 = (1,1,0,0,0)$, $A_3 = (1,0,1,0,1)$, $A_4 = (1,1,0,1,1)$, $A_5 = (1,0,0,0,0)$.

(d) $A_1 = (1,-1)$, $A_2 = (0,1)$, $A_3 = (1,1)$.

## A formula for $A^{-1}$

We know that an $n$ by $n$ matrix $A$ has an inverse if and only if it has rank $n$, and we know that $A$ has rank $n$ if and only if $\det A \neq 0$. Now we derive a formula for the inverse that involves determinants directly.

We begin with a lemma about the evaluation of determinants.

Lemma 18. Given the row matrix $[a_1 \ \dots \ a_n]$, let us define a function $f$ of $(n-1)$ by $(n-1)$ matrices $B$ by the formula

$$f(B) = \det \left[\begin{array}{c|c|c} a_1 \ \dots & a_j & \dots \ a_n \\ \hline & 0 & \\ B_1 & \vdots & B_2 \\ & 0 & \end{array}\right]$$

where $B_1$ consists of the first $j-1$ columns of $B$, and $B_2$ consists of the remainder of $B$. Then

$$f(B) = (-1)^{j+1} a_j \cdot \det B.$$

Proof. You can readily check that $f$ satisfies properties (1)-(3) of the determinant function. Hence $f(B) = f(I_{n-1}) \cdot \det B$. We compute

$$f(I_n) = \det \left[\begin{array}{c|c|c} a_1 \ \dots & a_j & \dots \ a_n \\ \hline I_{j-1} & 0 & \mathbf{0} \\ & \vdots & \\ \hline \mathbf{0} & 0 & I_{n-j} \end{array}\right]$$

where the large zeros stand for zero matrices of the appropriate size. A sequence of $j-1$ interchanges of adjacent rows gives us the equation

$$f(I_n) = (-1)^{j-1}\det\left[\begin{array}{c|c|c} I_{j-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & O \\ \hline a_1 \cdots & a_j & \cdots \quad a_n \\ \hline O & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{n-j} \end{array}\right].$$

One can apply elementary operations to this matrix, without changing the value of the determinant, to replace all of the entries $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ by zeros. Then the resulting matrix is in diagonal form. We conclude that

$$f(I_n) = (-1)^{j-1} a_j = (-1^{j+1} a_j . \square$$

Corollary 19. Consider an $n$ by $n$ matrix of the form

$$A = \begin{array}{cl} \begin{array}{c} \text{column } j \end{array} & \\ \left[\begin{array}{c|c|c} B_1 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & B_2 \\ \hline a_{i1} \cdots & a_{ij} & \cdots \quad a_{in} \\ \hline B_3 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & B_4 \end{array}\right] & \begin{matrix} \\ \leftarrow \text{row } i \\ \\ \end{matrix} \end{array}$$

where $B_1, \ldots, B_4$ are matrices of appropriate size. Then

$$\det A = (-1)^{i+j} a_{ij} \cdot \det \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

Proof. A sequence of $i-1$ interchanges of adjacent rows will bring the matrix $A$ to the form given in the preceding lemma. $\square$

Definition. In general, if $A$ is an $n$ by $n$ matrix, then the matrix of size $(n-1)$ by $(n-1)$ obtained by deleting the $i^{th}$ row and the $j^{th}$ column of $A$ is called the (i,j)-minor of $A$, and is denoted $A_{ij}$.

The preceding corollary can then be restated as follows:

Corollary 20. If all the entries in the $j^{th}$ column of $A$ are zero except for the entry $a_{ij}$ in row $i$, then $\det A = (-1)^{i+j} a_{ij} \cdot \det A_{ij}$.

The number $(-1)^{i+j} \det A_{ij}$ that appears in this corollary is also given a special name. It is called the (i,j)-cofactor of A. Note that the signs $(-1)^{i+j}$ follows the pattern

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ & & \cdots & & & \end{bmatrix}$$

Now we derive our formula for $A^{-1}$.

Theorem 21. Let $A$ be an $n$ by $n$ matrix with $\det A \neq 0$. If $A \cdot B = I_n$, then

$$b_{ij} = (-1)^{j+i} \det A_{ji}/\det A.$$

(That is, the entry of $B$ in row $i$ and column $j$ equals the $(j,i)$-cofactor of $A$, divided by $\det A$. This theorem says that you can compute $B$ by computing $\det A$ and the determinants of $n^2$ different $(n-1)$ by $(n-1)$ matrices. This is certainly not a practical procedure except in low dimensions!)

Proof. Let $X$ denote the $j^{th}$ column of $B$. Then $x_i = b_{ij}$. Because $A \cdot B = I_n$, the column matrix $X$ satisfies the equation

$$A \cdot X = (j^{th} \text{ column of } I_n) = E_j .$$

(Here $E_j$ is the column matrix consisting of zeros except for an entry of $1$ in row $j$.) Furthermore, if $A_i$ denote the $i^{th}$ column of $A$, then because $A \cdot I_n = A$, we have the equation

$$A \cdot (i^{th} \text{ column of } I_n) = A \cdot E_i = A_i .$$

Now we introduce a couple of weird matrices for reasons that will become clear. Using the two preceding equations, we put them together to get the following matrix equation:

$$(*) \qquad A \cdot [E_1 \ldots E_{i-1} \;\; X \;\; E_{i+1} \ldots \;\; E_n] \;=\; [A_1 \ldots A_{i-1} \;\; E_j \;\; A_{i+1} \ldots A_n].$$

It turns out that when we take determinants of both sides of this equation, we get exactly the equation of our theorem! First, we show that

$$\det [E_1 \ldots E_{i-1} \;\; X \;\; E_{i+1} \ldots E_n] = x_i .$$

Written out in full, this equation states that

$$\det \left[\begin{array}{c|c|c} & x_1 & \\ I_{i-1} & \vdots & 0 \\ \hline 0 \ldots 0 & x_i & 0 \ldots 0 \\ \hline & \vdots & \\ 0 & x_n & I_{n-i} \end{array}\right] = x_i .$$

If $x_i = 0$, this equation holds because the matrix has a zero row. If $x_i \neq 0$, we can by elementary operations replace all the entries above and beneath $x_i$ in its column by zeros. The resulting matrix will be in diagonal form, and its determinant will be $x_i$.

Thus the determinant of the left side of equation (*) equals $(\det A) \cdot x_i$, which equals $(\det A) \cdot b_{ij}$. We now compute the determinant of the right side of equation (*). Corollary 20 applies, because the $i^{th}$ column of this matrix consists of zeros except for an entry of 1 in row $j$. Thus the right side of (*) equals $(-1)^{j+i}$ times the determinant of the matrix obtained by deleting row $j$ and column $i$. This is exactly the same matrix as we would obtain by deleting row $j$ and column $i$ of $A$. Hence the right side of (*) equals $(-1)^{j+i} \det A_{ji}$, and our theorem is proved. $\square$

Remark 1. If $A$ is a matrix with general entry $a_{ij}$ in row $i$ and column $j$, then the transpose of $A$ (denoted $A^{tr}$) is the matrix whose entry in row $i$ and column $j$ is $a_{ji}$.

Thus if $A$ has size $k$ by $n$, then $A^{tr}$ has size $n$ by $k$; it can be pictured as the matrix obtained by flipping $A$ around the line $y = -x$. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{tr} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

Of course, if $A$ is square, then the transpose of $A$ has the same dimensions as $A$.

Using this terminology, the theorem just proved says that the inverse of $A$ can be computed by the following four-step process:

(1) Form the matrix whose entry in row $i$ and column $j$ is the number $\det A_{ij}$. (This is called the matrix of minor determinants.)

(2) Prefix the sign $(-1)^{i+j}$ to the entry in row $i$ and column $j$, for each entry of the matrix. (This is called the matrix of cofactors.)

(3) Transpose the resulting matrix.

(4) Divide each entry of the matrix by $\det A$.

In short, this theorem says that

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^{tr}.$$

This formula for $A^{-1}$ is used for computational purposes only for 2 by 2 or 3 by 3 matrices; the work simply gets too great otherwise. But it is important for theoretical purposes. For instance, if the entries of $A$

are continuous functions of a parameter $t$, this theorem tells us that the entries of $A^{-1}$ are also continuous functions of $t$, provided $\det A$ is never zero.

Remark 2. This formula does have one practical consequence of great importance. It tells us that if $\det A$ is small as compared with the entries of $A$, then a small change in the entries of $A$ is likely to result in a large change in the computed entries of $A^{-1}$. This means, in an engineering problem, that a small error in calculating $A$ (even round-off error) may result in a gross error in the calculated value of $A^{-1}$. A matrix for which $\det A$ is relatively small is said to be ill-conditioned. If such a matrix arises in practice, one usually tries to reformulate the problem to avoid dealing with such a matrix.

Exercises

1. Use the formula for $A^{-1}$ to find the inverses of the following matrices, assuming the usual definition of the determinant in low dimensions.

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assuming $ad - bc \neq 0$.

(b) $\begin{bmatrix} a & b & 0 \\ 0 & c & d \\ 0 & 0 & e \end{bmatrix}$, assuming $ace \neq 0$.

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

2. Let $A$ be a square matrix all of whose entries are integers. Show that if $\det A = \pm 1$, then all the entries of $A^{-1}$ are integers.

3. Consider the matrices A,B,C,D,E of p. A.23. Which of these matrices have inverses?

4. Consider the following matrix function:

$$A(t) = \begin{bmatrix} t & t^2 & t^3 \\ 0 & 1 & t \\ 2 & 0 & t \end{bmatrix}$$

For what values of $t$ does $A^{-1}$ exist? Give a formula for $A^{-1}$ in terms of $t$.

5. Show that the conclusion of Theorem 2.0 holds if $A$ has an entry of $a_{ij}$ in row $i$ and column $j$, and all the other entries in row $i$ equal $0$.

*6. <u>Theorem</u> Let $A$, $B$, $C$ be matrices of size $k$ by $k$, and $m$ by $k$, and $m$ by $m$, respectively. Then

$$\det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = (\det A)(\det C).$$

(Here $0$ is the zero matrix of appropriate size.)

<u>Proof.</u> Let $B$ and $C$ be fixed. For each $k$ by $k$ matrix $A$, define

$$f(A) = \det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}.$$

(a) Show $f$ satisfies the elementary row properties of the determinant function.

(b) Use Exercise 5 to show that $f(I_k) = \det C$.

(c) Complete the proof.

Construction of the determinant when $n \leq 3$.

The actual definition of the determinant function is the least interesting part of this entire discussion. The situation is similar to the situation with respect to the functions $\sin x$, $\cos x$, and $e^x$. You will recall that their actual definitions (as limits of power series) were not nearly as interesting as the properties we derived from simple basic assumptions about them.

We first consider the case where $n \leq 3$, which is doubtless familiar to you. This case is in fact all we shall need for our applications to calculus.

We begin with a lemma:

Lemma 21. Let $f(A)$ be a real-valued function of $n$ by $n$ matrices. Suppose that:

(i) Exchanging any two rows of $A$ changes the value of $f$ by a factor of $-1$.

(ii) For each $i$, $f$ is linear as a function of the $i^{th}$ row.

Then $f$ satisfies the elementary row properties of the determinant function.

Proof. By hypothesis, $f$ satisfies the first elementary row property. We check the other two.

Let $A_1,\ldots,A_n$ be the rows of $A$. To say that $f$ is linear as a function of row $i$ alone is to say that (when $f$ is written as a function of the rows of $A$):

$$(*)\quad f(A_1,\ldots, cX + dY, \ldots ,A_n) = cf(A_1,\ldots,X,\ldots,A_n) + df(A_1,\ldots,Y,\ldots,A_n),$$

where $cX + dY$ and $X$ and $Y$ appear in the $i^{th}$ component.

The special case $d = 0$ tells us that multiplying the $i^{th}$ row of $A$ by $c$ has the effect of multiplying the value of $f$ by $c$.

We now consider the third type of elementary operation. Suppose that $B$ is the matrix obtained by replacing row $i$ of $A$ by itself plus $c$ times row $j$. We then compute (assuming $j > i$ for convenience in notation),

$$f(B) = f(A_1,\ldots,\underset{\uparrow_{i\text{th}}}{A_i + cA_j},\ldots,\underset{\uparrow_{j\text{th}}}{A_j},\ldots,A_n)$$

$$= f(A_1,\ldots,\underset{\uparrow_{i\text{th}}}{A_i},\ldots,\underset{\uparrow_{j\text{th}}}{A_j},\ldots,A_n) + c\, f(A_1,\ldots,\underset{\uparrow_{i\text{th}}}{A_j},\ldots,\underset{\uparrow_{j\text{th}}}{A_j},\ldots,A_n).$$

The second term vanishes, since two rows are the same. (Exchanging them does not change the matrix, but by Step 1 it changes the value of $f$ by a factor of $-1$.) $\square$

Definition. We define

$$\det\begin{bmatrix} a \end{bmatrix} = a.$$

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1.$$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1\cdot\det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2\cdot\det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3\cdot\det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

Theorem 22. The preceding definitions satisfy the four conditions of the determinant function.

Proof. The fact that the determinant of the identity matrix is 1 follows by direct computation. It then suffices to check that (i) and (ii) of the preceding theorem hold .

In the 2 by 2 case, exchanging rows leads to the determinant $b_1a_2 - b_2a_1$, which is the negative of what is given.

In the 3 by 3 case, the fact that exchanging the last two rows changes the sign of the determinant follows from the 2 by 2 case. The fact that exchanging the first two rows also changes the sign follows similarly if we rewrite the formula defining the determinant in the form

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\cdot c_3 \;-\; \det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\cdot c_2 \;+\; \det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\cdot c_1 \;.$$

Finally, exchanging rows 1 and 3 can be accomplished by three exchanges of adjacent rows [ namely, $(A,B,C) \to (A,C,B) \to (C,A,B) \to (C,B,A)$ ], so it changes the sign of the determinant.

To check (ii) is easy. Consider the 3 by 3 case, for example. We know that any function of the form

$$f(X) = [\,a \quad b \quad c\,]\cdot X = ax_1 + bx_2 + cx_3$$

is linear, where $X$ is a vector in $V_3$. The function

$$f(X) = \det\begin{bmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

has this form, where the coefficients a, b, and c involve the constants $b_i$ and $c_j$. Hence $f$ is linear as a function of the first row. The "row-exchange property" then implies that $f$ is linear as a function of each of the other rows. □

Exercise

*1. Let us define

$$\det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} = a_1\cdot\det\begin{bmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{bmatrix} - a_2\cdot\det\begin{bmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{bmatrix}$$

$$+ a_3\cdot\det\begin{bmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{bmatrix} - a_4\cdot\det\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}.$$

(a) Show that $\det I_4 = 1$.

(b) Show that exchanging any two of the last three rows changes the sign of the determinant.

(c) Show that exchanging the first two rows changes the sign. [Hint: Write the expression as a sum of terms involving $\det\begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}$. ]

(d) Show that exchanging any two rows changes the sign.

(e) Show that det is linear as a function of the first row.

(f) Conclude that det is linear as a function of the $i^{th}$ row.

(g) Conclude that this formula satisfies all the properties of the determinant function.

**Construction of the Determinant Function** in general. Suppose we take the positive integers 1, 2, . . . , $k$ and write them down in some arbitrary order, say $j_1, j_2, \ldots, j_k$. This new ordering is called a *permutation* of these integers. For each integer $j_i$ in this ordering, let us count how many integers *follow* it in this ordering, but *precede* it in the natural ordering 1, 2, . . . , $k$. This number is called the *number of inversions caused by* the integer $j_i$. If we determine this number for each integer $j_i$ in the ordering and add the results together, the number we get is called the total number of inversions which occur in this ordering. If the number is odd, we say the permutation is an *odd permutation*; if the number is even, we say it is an *even permutation.*

For example, consider the following reordering of the integers between 1 and 6:

$$2, 5, 1, 3, 6, 4.$$

If we count up the inversions, we see that the integer 2 causes one inversion, 5 causes three inversions, 1 and 3 cause no inversions, 6 causes one inversion, and 4 causes none. The sum is five, so the permutation is odd.

If a permutation is odd, we say the *sign* of that permutation is $-$; if it is even, we say its sign is $+$. A useful fact about the sign of a permutation is the following:

> **Theorem 22.** If we interchange two adjacent elements of a permutation, we change the sign of the permutation.

*Proof.* Let us suppose the elements $j_i$ and $j_{i+1}$ of the permutation $j_1, \ldots, j_i, j_{i+1}, \ldots, j_k$ are the two we interchange, obtaining the permutation

$$j_1, \ldots, j_{i+1}, j_i, \ldots, j_k.$$

The number of inversions caused by the integers $j_1, \ldots, j_{i-1}$ clearly is the same in the new permutation as in the old one, and so is the number of inversions caused by $j_{i+2}, \ldots, j_k$. It remains to compare the number of inversions caused by $j_{i+1}$ and by $j_i$ in the two permutations.

*Case* I: $j_i$ precedes $j_{i+1}$ in the natural ordering $1, \ldots, k$. In this case, the number of inversions caused by $j_i$ is the same in both permutations, but the number of inversions caused by $j_{i+1}$ is one larger in the second permutation than in the first, for $j_i$ follows $j_{i+1}$ in the second permutation, but not in the first. Hence the total number of inversions is increased by one.

*Case* II: $j_i$ follows $j_{i+1}$ in the natural ordering $1, \ldots, k$. In this case, the number of inversion caused by $j_{i+1}$ is the same in both permutations, but the number of inversions caused by $j_i$ is one less in the second permutation than in the first.

In either case the total number of inversions changes by one, so that the sign of the permutation changes. □

EXAMPLE. If we interchange the second and third elements of the permutation considered in the previous example, we obtain 2, 1, 5, 3, 6, 4, in which the total number of inversions is four, so the permutation is even.

**Definition.** Consider a $k$ by $k$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}.$$

Pick out one entry from each row of $A$; do this in such a way that these entries all lie in different columns of $A$. Take the product of these entries,

$$a_{1j_1}a_{2j_2}a_{3j_3}\cdots a_{kj_k},$$

and prefix a $\pm$ sign according as the permutation $j_1, \ldots, j_k$ is even or odd. (Note that we arrange the entries in the order of the rows they come from, and then we compute the sign of the resulting permutation of the column indices.)

If we write down *all possible* such expressions and add them together, the number we get is defined to be the *determinant* of $A$.

REMARK. We apply this definition to the general 2 by 2 matrix, and obtain the formula

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

If we apply it to a 3 by 3 matrix, we find that

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{matrix} + a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{matrix}$$

The formula for the determinant of a 4 by 4 matrix involves 24 terms, and for a 5 by 5 matrix it involves 120 terms; we will not write down these formulas. The reader will readily believe that the definition we have given is not very useful for computational purposes!

The definition is, however, very convenient for theoretical purposes.

Theorem 24. The determinant of the identity matrix is 1.

Proof. Every term in the expansion of $\det I_n$ has a factor of zero in it except for the term $a_{11}a_{22}\cdots a_{kk}$, and this term equals 1. □

Theorem 25. If $A'$ is obtained from $A$ by interchanging rows $i$ and $i+1$, then $\det A' = -\det A$.

Proof. Note that each term in the expansion of $\det A'$ also appears in the expansion of $\det A$, because we make *all possible* choices of one entry from each row and column when we write down this expansion. The only thing we have to do is to compare what signs this term has when it appears in the two expansions.

Let $a_{1j_1} \cdots a_{ij_i}a_{i+1,j_{i+1}} \cdots a_{kj_k}$ be a term in the expansion of $\det A$. If we look at the corresponding term in the expansion of $\det A'$, we see that we have the same factors, *but they are arranged differently*. For to compute the sign of this term, we agreed to arrange the entries in the order of the rows they came from, and then to take the sign of the corresponding permutation of the column indices. Thus in the expansion of $\det A'$, this term will appear as

$$a_{1j_1} \cdots a_{i+1,j_{i+1}}a_{i,j_i} \cdots a_{kj_k}.$$

The permutation of the column indices here is the same as above except that elements $j_i$ and $j_{i+1}$ have been interchanged. By Theorem 8.4, this means that this term appears in the expansion of $\det A'$ with the sign opposite to its sign in the expansion of $\det A$.

Since this result holds for each term in the expansion of $\det A'$, we have $\det A' = -\det A$. □

Theorem 26. The function det is linear as a function of the $i^{th}$ row.

Proof. Suppose we take the constant matrix $A$, and replace its $i^{th}$ row by the row vector $[x_1 \ \ldots \ x_k]$. When we take the determinant of this new matrix, each term in the expression equals a constant times $x_j$, for some $j$. (This happens because in forming this term, we picked out exactly one entry from each row of $A$.) Thus this function is a linear combination of the components $x_i$; that is, it has the form

$[c_1 \ \ldots \ c_k] \cdot X$, for some constants $c_i$. □

Exercises

1. Use Theorem 25 to show that exchanging any two rows of $A$ changes the sign of the determinant.

2. Consider the term $a_{1j_1} \cdot a_{2j_2} \cdots a_{kj_k}$ in the definition of the determinant. (The integers $j_1, j_2, \ldots, j_k$ are distinct.) Suppose we arrange the factors in this term in the order of their column indices, obtaining an expression of the form

$$a_{i_1 1} \cdot a_{i_2 2} \cdots a_{i_k k} .$$

Show that the sign of the permutation $i_1, i_2, \ldots, i_k$ equals the sign of the permutation $j_1, j_2, \ldots, j_k$.

Conclude that $\det A^{tr} = \det A$ in general.

3. Let $A$ be an $n$ by $n$ matrix, with general entry $a_{ij}$ in row $i$ and column $j$. Let $m$ be a fixed index. Show that

$$\det A = \sum_{j=1}^{n} a_{mj} \cdot (-1)^{m+j} \det A_{mj} .$$

Here $A_{mj}$ denotes , as usual, the $(m,j)$-minor of $A$. This formula is called the "formula for expanding $\det A$ according to the cofactors of the $m^{th}$ row." [Hint: Write the $m^{th}$ row as the sum of $n$ vectors, each of which has a single non-zero component. Then use the fact that the determinant function is linear as a function of the $m^{th}$ row.]

## The cross-product in $V_3$

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are vectors in $V_3$, we define their cross product to be the vector

$$A \times B = \left(\det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}, -\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}, \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right).$$

We shall describe the geometric significance of this product shortly. But first, we prove some properties of the cross product:

Theorem 27. For all vectors $A$, $B$ in $V_3$, we have

(a) $B \times A = -A \times B$.

(b) $A \times (B + C) = A \times B + A \times C$,

$(B + C) \times A = B \times A + C \times A$.

(c) $(cA) \times B = c(A \times B) = A \times (cB)$.

(d) $A \times B$ is orthogonal to both $A$ and $B$.

(e) $\|A \times B\|^2 = \|A\|^2 \cdot \|B\|^2 - (A \cdot B)^2$.

Proof. (a) follows because exchanging two rows of a determinant changes the sign; and (b) and (c) follows because the determinant is linear as a function of each row separately. To prove (d), we note that if $C = (c_1, c_2, c_3)$, then

$$C \cdot (A \times B) = \det\begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix},$$

by definition of the determinant. It follows that $A \cdot (A \times B) = B \cdot (A \times B) = 0$ because the determinant vanishes if two rows are equal. The only proof that requires some work is (e). For this, we recall that $(a + b)^2 = a^2 + b^2 + 2ab$, and $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. Equation (e) can be written in the form

$$(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2 + a_3b_3)^2 =$$
$$(a_1^2 + a_2^2 + a_3^2)\ (b_1^2 + b_2^2 + b_3^2)\ .$$

We first take the squared terms on the left side and show they equal the right side. Then we take the "mixed" terms on the left side and show they equal zero. The squared terms on the left side are

$$(a_2b_3)^2 + (a_3b_2)^2 + (a_1b_3)^2 + (a_3b_1)^2 + (a_1b_2)^2 + (a_2b_1)^2 + (a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2,$$

which equals the right side,

$$\sum_{i,j=1}^{3} (a_ib_j)^2 \ .$$

The mixed terms on the left side are

$$-2a_2b_3a_3b_2 - 2a_1b_3a_3b_1 - 2a_1b_2a_2b_1 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3 = 0. \ \square$$

In the process of proving the previous theorem, we proved also the following:

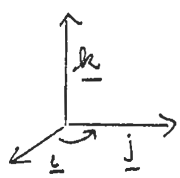
<u>Theorem</u> 28. Given A, B, C , we have $A\cdot(B\times C) = (A\times B)\cdot C$.

<u>Proof.</u> This follows from the fact that

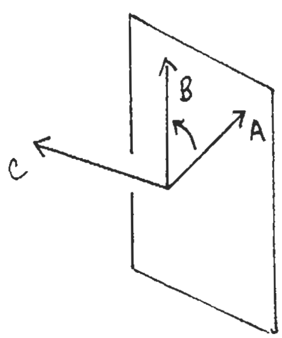
$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det\begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \ \square$$

<u>Definition</u>. The ordered 3-tuple of independent vectors (A,B,C) of vectors of $V_3$ is called a <u>positive</u> triple if $A\cdot(B\times C) > 0$. Otherwise, it is called a <u>negative</u> triple. A positive triple is sometimes said to be a <u>right-handed</u> triple, and a negative one is said to be <u>left-handed</u>.

The reason for this terminology is the following: (1) the triple $(\underline{i}, \underline{j}, \underline{k})$ is a positive triple, since $\underline{i}\cdot(\underline{j}\times\underline{k}) = \det I_3 = 1$, and (2) if we draw the vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$ in $V_3$ in the usual way, and if one curls the fingers of one's right hand in the direction from the first to the second, then one's thumb points in the direction of the third.




Furthermore, if one now moves the vectors around in $V_3$, perhaps changing their lengths and the angles between them, but <u>never letting them become dependent,</u> and if one moves one's right hand around correspondingly, then the fingers still correspond to the new triple (A,B,C) in the same way, and this new triple is still a positive triple, since the determinant cannot have changed sign while the vectors moved around. (Since they did not become dependent, the determinant did not vanish.)




<u>Theorem</u> 29. Let A and B be vectors in $V_3$. If A and B are dependent, then $A\times B = \underline{0}$. Otherwise, $A\times B$ is the unique vector orthogonal to both A and B having length $\|A\|\,\|B\|\sin\theta$ (where $\theta$ is the angle between A and B), such that the triple $(A,B,A\times B)$ forms a positive (i.e., right-handed) triple.

Proof. We know that $A \times B$ is orthogonal to both $A$ and $B$. We also have

$$\|A \times B\|^2 = \|A\|^2 \cdot \|B\|^2 - (A \cdot B)^2$$
$$= \|A\|^2 \cdot \|B\|^2 (1 - \cos^2 \theta) = \|A\|^2 \|B\|^2 \sin^2 \theta .$$

Finally, if $C = A \times B$, then $(A,B,C)$ is a positive triple, since

$$A \cdot (B \times C) = (A \times B) \cdot C = (A \times B) \cdot (A \times B) = \|A \times B\|^2 > 0 . \square$$

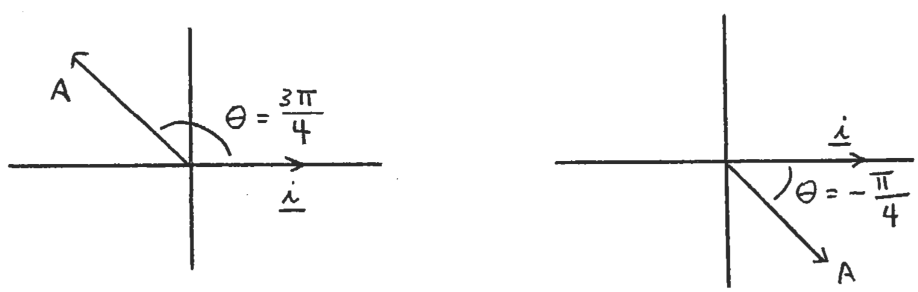
# Polar coordinates

Let $A = (a,b)$ be a point of $V_2$ different from $\underline{0}$. We wish to define what we mean by a "polar angle" for A. The idea is that it should be the angle between the vector A and the unit vector $\underline{i} = (1,0)$. But we also wish to choose it so its value reflects whether A lies in the upper or lower half–plane. So we make the following definition:

Definition. Given $A = (a,b) \neq \underline{0}$. We define the number

$$(*) \qquad \theta = \pm \arccos (A \cdot \underline{i}/\|A\|)$$

to be a polar angle for A, where the sign in this equation is specified to be + if $b > 0$, and to be – if $b < 0$. Any number of the form $2m\pi + \theta$ is also defined to be a polar angle for A.




If $b = 0$, the sign in this equation is not determined, but that does not matter. For if $A = (a,0)$ where $a > 0$, then $\arccos (A \cdot i/\|A\|) = \arccos 1 = 0$, so the sign does not matter. And if $A = (-a,0)$ where $a > 0$, then $\arccos (A \cdot \underline{i}/\|A\|) = \arccos (-1) = \pi$. Since the two numbers $+\pi$ and $-\pi$ differ by a multiple of $2\pi$, the sign does not matter, for since one is a polar angle for A, so is the other.

Note: The polar angle $\theta$ for A is uniquely determined if we require $-\pi < \theta \leq \pi$. But that is a rather artificial restriction.

Theorem. Let $A = (a,b) \neq \underline{0}$ be a point of $V_2$. Let $r = (a^2+b^2)^{1/2} = \|A\|$; let $\theta$ be a polar angle for A. Then

$$A = (r \cos \theta, r \sin \theta).$$

Proof. If $A = (a,0)$ with $a > 0$, then $r = a$ and $\theta = 0 + 2m\pi$; hence

$$r \cos \theta = a \quad \text{and} \quad r \sin \theta = 0.$$

If $A = (-a,0)$ with $a > 0$, then $r = a$ and $\theta = \pi + 2m\pi$, so that

$$r \cos \theta = -a \quad \text{and} \quad r \sin \theta = 0.$$

Finally, suppose $A = (a,b)$ with $b \neq 0$. Then $A \cdot \underline{i}/\|A\| = a/r$, so that

$$\theta = 2m\pi \pm \arccos(a/r).$$

Then

$$a/r = \cos(\pm(\theta - 2m\pi)) = \cos \theta, \quad \text{or} \quad a = r \cos \theta.$$

Furthermore,

$$b^2 = r^2 - a^2 = r^2(1 - \cos^2\theta) = r^2 \sin^2\theta,$$

so

$$b = \pm r \sin \theta.$$

We show that in fact $b = r \sin \theta$. For if $b > 0$, then $\theta = 2m\pi + \arccos(a/r)$, so that

$$2m\pi < \theta < 2m\pi + \pi$$

and $\sin \theta$ is positive. Because $b$, $r$, and $\sin \theta$ are all positive, we must have $b = r \sin \theta$ rather than $b = -r \sin \theta$.

On the other hand, if $b < 0$, then $0 = 2m\pi - \arccos(a/r)$, so that

$$2m\pi - \pi < \theta < 2m\pi$$

and $\sin \theta$ is negative. Since $r$ is positive, and $b$ and $\sin \theta$ are negative, we must have $b = r \sin \theta$ rather than $b = -r \sin \theta$. □

## Planetary Motion

In the text, Apostol shows how Kepler's three (empirical) laws of planetary motion can be deduced from the following two laws:

(1) Newton's second law of motion: $\underline{F} = m\underline{a}$.

(2) Newton's law of universal gravitation:

$$\|\underline{F}\| = G \frac{mM}{r^2}.$$

Here m, M are the masses of the two objects, r is the distance between them, and G is a universal constant.

Here we show (essentially) the reverse—how Newton's laws can be deduced from Kepler's.

More precisely, suppose a planet P of mass m moves in the xy plane with the sun whose mass is M at the origin. Newton's laws tell us that the acceleration of P is given by the equation

$$\underline{a} = \frac{1}{m}\underline{F} = \frac{1}{m}\left[-G\frac{mM}{r^2}\right]\underline{u}_r = -\frac{GM}{r^2}\underline{u}_r.$$

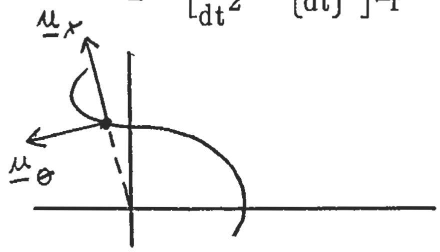
That is, Newton's laws tell us that there is a number $\lambda$ such that

$$\underline{a} = -\frac{\lambda}{r^2}\underline{u}_r,$$

and that $\lambda$ is the <u>same</u> for all planets in the solar system. (One needs to consider other systems to see that $\lambda$ involves the mass of the sun.)

This is what we shall prove. We use the formula for acceleration in polar coordinates (Apostol, p. 542):

$$\underline{a} = \left[\frac{d^2r}{dt^2} - r\left[\frac{d\theta}{dt}\right]^2\right]\underline{u}_r + \left[2\,\frac{dr}{dt}\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right]\underline{u}_\theta$$




We also use some facts about area that we shall not actually prove until Units VI and VII of this course.

<u>Theorem. Suppose a planet P moves in the xy plane with the sun at the origin.</u>

(a) <u>Kepler's second law implies that the acceleration is radial.</u>

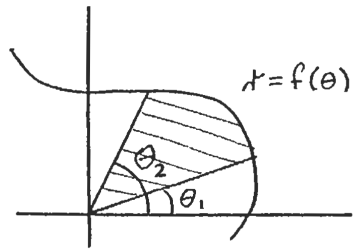
(b) <u>Kepler's first and second laws imply that</u>

$$\underline{a} = -\frac{\lambda_P}{r^2}\underline{u}_r,$$

<u>where $\lambda_P$ is a number that may depend on the particular planet</u> P.

(c) <u>Kepler's three laws imply that $\lambda_P$ is the same for all planets.</u>

<u>Proof</u>. (a) We use the following formula for the area swept out by the radial vector as the planet moves from polar angle $\theta_1$ to polar angle $\theta_2$:




$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta.$$

Here it is assumed the curve is specified by giving r as a function of $\theta$.

Now in our present case both $\theta$ and r are functions of time t. Hence the area swept out as time goes from $t_0$ to t is (by the substitution rule) given by

$$A(t) = \int_{t_0}^{t} \left[\frac{1}{2} r^2 \frac{d\theta}{dt}\right] dt.$$

Differentiating, we have $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$, which is constant by Kepler's second law. That is,

(*) $$\boxed{2\frac{dA}{dt} = r^2 \frac{d\theta}{dt} = K}$$

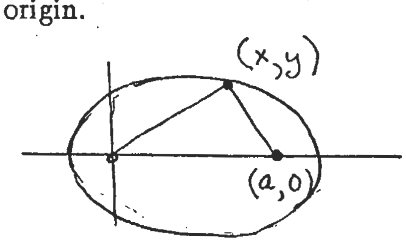
for some K.

Differentiating, we have

$$2r \frac{dr}{dt}\frac{d\theta}{dt} + r^2 \frac{d^2\theta}{dt^2} = 0.$$

The left side of this equation is just the transverse component (the $\mu_\theta$ component) of $\underline{a}$! Hence $\underline{a}$ is radial.

(b) To apply Kepler's first law, we need the equation of an ellipse with focus at the origin.




We put the other focus at (a,0), and use the fact that an ellipse is the locus of all points (x,y) the sum of whose distances from (0,0) and (a,0) is a constant b > a.

The algebra is routine:

$$\sqrt{x^2 + y^2} + \sqrt{(x-a)^2 + y^2} = b,$$

or

$$r + \sqrt{r^2 - 2a(r\cos\theta) + a^2} = b,$$

$$r^2 - 2a(r\cos\theta) + a^2 = (b-r)^2 = b^2 - 2br + r^2,$$

$$2br - 2ar\cos\theta = b^2 - a^2,$$

$$r = \frac{(b^2-a^2)/2b}{1 - \frac{a}{b}\cos\theta},$$

(**) $$\boxed{r = \frac{c}{1 - e\cos\theta},\ \text{where } c = \frac{b^2 - a^2}{2b} \text{ and } e = a/b.}$$

(The number $e$ is called the <u>eccentricity</u> of the ellipse, by the way.) Now we compute the radial component of acceleration, which is

$$\left[\frac{d^2r}{dt^2} - r\left[\frac{d\theta}{dt}\right]^2\right].$$

Differentiating (**), we compute

$$\frac{dr}{dt} = c\left[\frac{-1}{(1-e\cos\theta)^2}(e\sin\theta)\frac{d\theta}{dt}\right].$$

Simplifying,

$$\frac{dr}{dt} = \frac{1}{c}(-1)r^2(e\sin\theta)\frac{d\theta}{dt}.$$

Then using (*) from p. B60, we have

$$\frac{dr}{dt} = \frac{1}{c}(e\sin\theta)K.$$

Differentiating again, we have

$$\frac{d^2r}{dt^2} = -\frac{1}{c}(e\cos\theta)\frac{d\theta}{dt}K,$$

or

$$\frac{d^2r}{dt^2} = -\frac{1}{c}(e\cos\theta)\left[\frac{K}{r^2}\right]K, \text{ using (*) to get rid of } d\theta/dt.$$

Similarly,

$$-r\left[\frac{d\theta}{dt}\right]^2 = -r\left[\frac{K}{r^2}\right]^2 \text{ using (*) again to get rid of } d\theta/dt.$$

Hence the radial component of acceleration is (adding these equations)

$$-\frac{1}{c}(e\cos\theta)\frac{K^2}{r^2} - \frac{K^2}{r^3} = -\frac{K^2}{r^2}\left[\frac{e\ \cos\ \theta}{c} + \frac{1}{r}\right]$$

$$= -\frac{K^2}{r^2}\left[\frac{e\ \cos\ \theta}{c} + \frac{1-e\ \cos\ \theta}{c}\right]$$

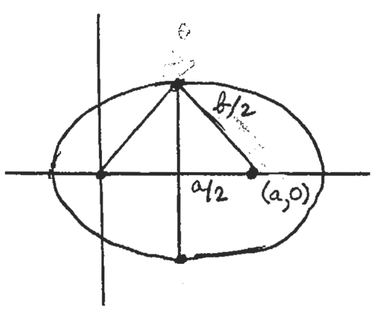
$$= -\left[\frac{K^2}{c}\right]\frac{1}{r^2}.$$

Thus, as desired,

(***) $$\boxed{\underline{a} = -\frac{\lambda_P}{r^2}\underline{\mu}_r, \quad \text{where} \quad \lambda_P = \frac{K^2}{c}.}$$

(c) To apply Kepler's third law, we need a formula for the area of an ellipse, which will be proved later, in Unit VII. It is

$$\text{Area} = \pi\frac{(\text{major axis})}{2}\frac{(\text{minor axis})}{2}.$$




The minor axis is easily determined to be given by:

$$\text{minor axis} = 2\sqrt{b^2/4-a^2/4} = \sqrt{b^2-a^2}.$$

It is also easy to see that

$$\text{major axis} = b.$$

Now we can apply Kepler's third law. Since area is being swept out at the constant rate $\frac{1}{2}K$, we know that (since the period is the time it takes to sweep out the entire area),

$$\text{Area} = (\tfrac{1}{2}K)(\text{Period}).$$

Kepler's third law states that the following number is the same for all planets:

$$\frac{(\text{Period})^2}{(\text{major axis})^3} = \frac{4(\text{Area})^2/K^2}{(\text{major axis})^3}$$

$$= \frac{4}{16}\,\frac{\pi^2(\text{major axis})^2(\text{minor axis})^2/K^2}{(\text{major axis})^3}$$

$$= \frac{\pi^2}{4}\,\frac{(\text{minor axis})^2}{(\text{major axis})}\,\frac{1}{K^2}$$

$$= \frac{\pi^2}{4}\,\frac{(b^2-a^2)}{b}\,\frac{1}{K^2}$$

$$= \frac{\pi^2}{2}\left[\frac{c}{K^2}\right] \quad \text{by (**)}$$

$$= \frac{\pi^2}{2}\,\frac{1}{\lambda_P} \quad \text{by (***)}.$$

Thus the constant $\lambda_P$ is the same for all planets. □

## SUPPLEMENTARY EXERCISES FOR UNIT III

(1) Let L be a line in $V_n$ with direction vector A; let P be a point not on L. Show that the point X on the line L closest to P satisfies the condition that $X - P$ is perpendicular to A.

(2) Find parametric equations for the curve C consisting of all points of $V_2$ equidistant from the point P = (0,1) and the line y = −1. If X is any point of C, show that the tangent vector to C at X makes equal angles with the vector $X - P$ and the vector $\vec{j}$. (This is the reflection property of the parabola.)

(3) Consider the curve $f(t) = (t, t\cos(\pi/t))$ for $0 < t \leq 1$,

$$= (0,0) \qquad \text{for } t = 0.$$

Then f is continuous. Let P be the partition

$$P = \{0, 1/n, 1/(n-1), \ldots, 1/3, 1/2, 1\}.$$

Draw a picture of the inscribed polygon $\pi(P)$ in the case n = 5. Show that in general, $\pi(P)$ has length

$$|\pi(P)| \geq 1 + 2(1/2 + 1/3 + \ldots + 1/n).$$

Conclude that f is not rectifiable.

(4) Let $\underline{u}$ be a fixed unit vector. A particle moves in $V_n$ in such a way that its position vector $\underline{r}(t)$ satisfies the equation $\underline{r}(t)\cdot\underline{u} = 5t^3$ for all t, and its velocity vector makes a constant angle $\theta$ with $\underline{u}$, where $0 < \theta < \pi/2$.

(a) Show that $\|\underline{v}\| = 15t^2/\cos\theta$.

(b) Compute the dot product $\underline{a}(t)\cdot\underline{v}(t)$ in terms of t and $\theta$.

(5) A particle moves in 3—space so as to trace out a curve of constant curvature $\kappa = 3$. Its speed at time t is $e^{2t}$. Find $\|\underline{a}(t)\|$, and find the angle between $\underline{v}$ and $\underline{a}$ at time t.

(6) Consider the curve given in polar coordinates by the equation $r = e^{-\theta}$ for $0 \le \theta \le 2\pi M$, where M is a positive integer. Find the length of this curve. What happens as M becomes arbitrarily large?

(7) (a) Derive the following formula, which can be used to compute the curvature of a curve in $R^n$:

$$(\underline{v}\cdot\underline{v})^2 \kappa \underline{N} = (\underline{v}\cdot\underline{v})\underline{a} - (\underline{a}\cdot\underline{v})\underline{v}.$$

(b) Find the curvature of the curve $\underline{r}(t) = (1+t,\ 3t,\ 2+t^2,\ 2t^2)$.

18.024 Multivariable Calculus with Theory
Spring 2011