SIXTH HOMEWORK

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1. (10 pts) Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$-function, and let C be the plane curve given by the equation $r = f(\theta)$ in polar coordinates. Show that the arc length of C is given by

$$s(\theta) = \int_{\alpha}^{\theta} \sqrt{f(\tau)^2 + (f'(\tau))^2}\, d\tau.$$

2. (5 pts) (3.1.18).
3. (10 pts) (3.1.26).
4. (10 pts) (3.1.30).
5. (5 pts) (3.1.32).
6. (10 pts) (3.2.7).
7. (10 pts) (3.2.12).
8. (10 pts) In this question, we prove the uniqueness statement of Theorem 2.5 on page 201 of the book. Let $\vec{r}_1\colon I \longrightarrow \mathbb{R}^3$ and $\vec{r}_2\colon I \longrightarrow \mathbb{R}^3$ be two smooth regular curves parametrised by arclength. Assume that $\kappa_1(s) = \kappa_2(s)$ and $\tau_1(s) = \tau_2(s)$, for every $s \in I$. Suppose that there is a point $a \in I$ where

$$\vec{r}_1(a) = \vec{r}_2(a), \quad \vec{T}_1(a) = \vec{T}_2(a), \quad \vec{N}_1(a) = \vec{N}_2(a), \quad \text{and} \quad \vec{B}_1(a) = \vec{B}_2(a).$$

(a) Show that the quantity

$$\|\vec{T}_1(s) - \vec{T}_2(s)\|^2 + \|\vec{N}_1(s) - \vec{N}_2(s)\|^2 + \|\vec{B}_1(s) - \vec{B}_2(s)\|^2,$$

is a constant function of s. (*Hint: Differentiate and use the Frenet-Serret formulae.*)

(b) Show that $\vec{r}_1(s) = \vec{r}_2(s)$, for all $s \in S$.

9. (10 pts) Let $\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ be the helix given by

$$\vec{r}(s) = a\cos\left(\frac{s}{c}\right)\hat{\imath} + a\sin\left(\frac{s}{c}\right)\hat{\jmath} + \frac{bs}{c}\hat{k},$$

where

$$c^2 = a^2 + b^2 \qquad a > 0 \qquad \text{and} \qquad c > 0.$$

(a) Show that $\vec{r}$ is parametrised by arclength.

(b) Find $\vec{T}(s)$, $\vec{N}(s)$ and $\vec{B}(s)$.

(c) Find $\kappa(s)$ and $\tau(s)$.

10. (10 pts) Suppose that $\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ is a smooth regular curve parametrised by arclength. Suppose that the curvature and the torsion are constant, that is, suppose that there are constants κ and τ such that $\kappa(s) = \kappa$ and $\tau(s) = \tau$. Prove that $\vec{r}$ is (congruent to) a helix.

11. (10 pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular curve. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{2}{3}\hat{\imath}+\frac{2}{3}\hat{\jmath}-\frac{1}{3}\hat{k}, \quad \vec{B}(a) = -\frac{1}{3}\hat{\imath}+\frac{2}{3}\hat{\jmath}+\frac{2}{3}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = -4\hat{\imath}+2\hat{\jmath}+5\hat{k}.$$

Find

(a) The normal vector $\vec{N}(a)$.

(b) The curvature $\kappa(a)$.

(c) The torsion $\tau(a)$.

Just for fun: Show that the implicit function theorem and the inverse function theorem are equivalent, that is, one can prove either result assuming the other result.