18.02 Practice Exam 4B

Problem 1. (10 points)

Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

(Give integrand and limits of integration, but *do not evaluate*.)

Problem 2. (20 points: 5, 5, 10)

a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates*.

b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate*.)

Problem 3. (20 points: 5, 15)

Let $\vec{F} = (2xy + z^3)\,\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 3xz^2 - 1)\,\hat{k}$.

a) Show that $\vec{F}$ is conservative.

b) Using a systematic method, find a potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$. Show your work, even if you can do it mentally.

Problem 4. (25 points: 15, 10)

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 2(1 - z)\,\hat{k}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) by direct calculation of $\iint_S \vec{F} \cdot \hat{n}\, dS$;

b) by computing the flux of $\vec{F}$ across a simpler surface and using the divergence theorem.

Problem 5. (25 points: 10, 8, 7)

Let $\vec{F} = -2xz\,\hat{\imath} + y^2\,\hat{k}$.

a) Calculate $\operatorname{curl} \vec{F}$.

b) Show that $\iint_R \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = 0$ for any finite portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector $\hat{n}$ pointing outward).

c) Show that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.