18.02: Practice Exam 3A

1. Let $(\bar{x}, \bar{y})$ be the center of mass of the triangle, with vertices at $(-2, 0)$, $(0, 1)$, $(2, 0)$ and uniform density $\delta = 1$.

a) Write an integral formula for $\bar{y}$. Do not evaluate the integral(s), but write explicitly the integrand and limits of integration.

b) Find $\bar{x}$.

2. Find the polar moment of inertia of the unit disk with density equal to the distance from the y-axis.

3. Let $\mathbf{F} = (ax^2y + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}$ be a vector field, where a and b are constants.

a) Find the values of a and b for which $\mathbf{F}$ is conservative.

b) For these values of a and b, find $f(x, y)$ such that $\mathbf{F} = \nabla f$.

c) Still using the values of a and b from part (a), compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi$.

4. For $\mathbf{F} = yx^3\mathbf{i} + y^2\mathbf{j}$ find $\int_C \mathbf{F} \cdot d\mathbf{r}$ on the portion of the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.

5. Consider the region R in the first quadrant bounded by the curves $y = x^2$, $y = x^2/5$, $xy = 2$, and $xy = 4$.

a) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

b) Find a double integral for the area of R in uv coordinates and evaluate it.

6. a) Let C be a simple closed curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\displaystyle\oint_C M dx$ as a double integral over R.

b) Find M so that $\displaystyle\oint_C M dx$ is the mass of R with density $\delta(x, y) = (x + y)^2$.

7. Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$. Travelling in a counterclockwise direction along the boundary C or R, call C_1 the portion of C that goes from $(0, 0)$ to $(0, 1)$, C_2 the portion that goes from $(1, 0)$ to $(1, 1)$ and C_3 the portion that goes from $(1, 1)$ to $(0, 0)$.

a) Find the total work of $\mathbf{F} = (1 + y^2)\mathbf{i}$ around the boundary C of R, in a counterclockwise direction.

b) Calculate the work of $\mathbf{F}$ along C_1 and C_2.

c) Use parts (a) and (b) to find the work along the third side C_3.

18.02 Practice Exam 3A Solutions

1. a) Area of triangle is base times height $= 2$, so $\bar{y} = \dfrac{1}{2}\displaystyle\int_0^1 \int_{2y-2}^{2-2y} y\; dxdy$

b) By symmetry $\bar{x} = 0$

2. $\delta = |x| = r|\cos\theta|$. $I_0 = \displaystyle\int\int_D r^2 \delta r dr d\theta =$

$$\int_0^{2\pi}\int_0^1 r^2 |r\cos\theta| r dr d\theta = 4\int_0^{\pi/2}\int_0^1 r^4 \cos\theta dr d\theta = 4\int_0^{\pi/2} \frac{1}{5}\cos\theta d\theta = \frac{4}{5}$$

3. a) $N_x = 6x^2 + by^2$, $M_y = ax^2 + 3y^2$. $N_x = M_y$ provided $a = 6$ and $b = 3$.

b) $f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + c(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + c'(y)$. Setting this equal to N, we have $2x^3 + 3xy^2 + c'(y) = 2x^3 + 3xy^2 + 2$ so $c'(y) = 2$ and $c = 2y$ (+constant). In all,

$$f = 2x^3y + xy^3 + x + 2y \quad (+\text{constant})$$

c) C starts at $(1, 0)$ and ends at $(-e^{\pi}, 0)$, so $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(-e^{\pi}, 0) - f(1, 0) = -e^{\pi} - 1$.

4. $\displaystyle\int_C yx^3dx + y^2dy = \int_0^1 x^2x^3dx + (x^2)^2(2xdx) = \int_0^1 3x^5dx = 1/2$

5. a) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = 3x^2/y$. Therefore,

$$dudv = (3x^2/y)dxdy = 3u\; dxdy \qquad \Longrightarrow \qquad dxdy = \frac{1}{3u}dudv$$

b) $\displaystyle\int_2^4\int_1^5 \frac{1}{3u}dudv = \int_2^4 \frac{1}{3}\ln 5\; dv = \frac{2}{3}\ln 5$

6. a) $\displaystyle\oint_C M dx = \int\int_R -M_y\; dA$

b) We want M such that $-M_y = (x + y)^2$. Use $M = -\dfrac{1}{3}(x + y)^3$

7. a) For $\mathbf{F}$, $M_y = 2y$ and $N_x = 0$, hence $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R -2y\; dA = \int_0^1\int_0^{x^3} -2y\; dydx = \int_0^1 -x^6dx = -\frac{1}{7}$.

b) For the work through C_1, we have $\mathbf{F}\cdot\mathbf{i} = 1 + y^2 = 1 + 0 = 1$. The length of C_1 is 1, so the total work through C_1 is 1.

The work through C_2 is zero because $\mathbf{F}\cdot\mathbf{j} = 0$.

c) $\displaystyle\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} - \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -\frac{1}{7} - 1 - 0 = -\frac{8}{7}$