Taylor's Series of $\dfrac{1}{1+x}$

Our next example is the Taylor's series for $\frac{1}{1+x}$; this series was first described by Isaac Newton. Remember the formula for the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{if } |x| < 1.$$

If we replace x by $-x$ we get:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad R = 1.$$

You may recall that the graph of this function has an infinite discontinuity at $x = -1$; this gives us an idea of what R might be. If we try to replace x by -1 we get something of the form $\infty = \infty$; the radius of convergence of this series is 1.

Instead of deriving this from the formula for the geometric series we could also have computed it using Taylor's formula. Try it!

Question: If you put in -1 for x the series diverges. If you put in 1, it looks like it would converge.

Answer: The graph of $y = \frac{1}{1+x}$ looks smooth at $x = 1$, but there is still a problem. If the series converges for $|x| < |a|$ and then diverges for $x = a$ the radius of convergence is a; that's it.

What happens if we plug $x = 1$ into the series $1 - x + x^2 - x^3 + \cdots$? Let's look at the partial sums $S_N = \sum_{n=0}^{N} a_n x^n$.

$$\begin{aligned} S_0 &= 1 \\ S_1 &= 0 \\ S_2 &= 1 \\ S_3 &= 0 \\ S_4 &= 1 \\ &\vdots \end{aligned}$$

Even though these don't go off to infinity, they still don't converge.