Introduction to Surface Area

We're going to move to three dimensions now to talk about surface area; we'll be doing a lot with surface area in multivariable calculus. If this starts to look too complicated, keep in mind that all we're doing is integrating infinitesimal pieces of simple, linear functions.

The only surface areas we'll compute in this class are surfaces of rotation. We'll start by rotating the parabola from our last example about the x-axis to get the trumpet shape shown in Figure 2. (Remember that we're only interested in the surface — the metal part of the trumpet — and not the interior.)

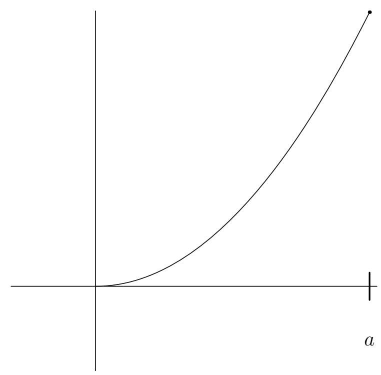

Figure 1: The parabola $y = x^2$.

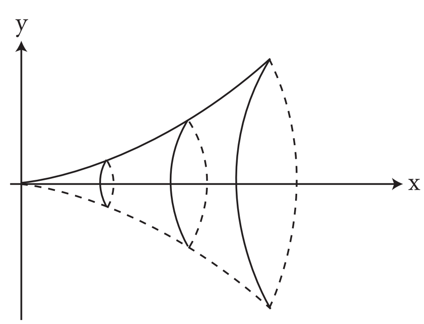

Figure 2: The parabola $y = x^2$ rotated about the x-axis.

We figure out the formula for surface area of a surface of rotation in much the same way we figured out the formula for volumes of revolution. Think of a small segment of arc length with length ds. If that segment is parallel to the x-axis, when you rotate it around the axis it sweeps out a shell shape. If the segment is tilted at an angle, then the surface swept out will have more area than a shell, proportional to the amount of tilt. The surface area swept out is proportional to the length ds of the segment.

In our example, the total surface area swept out by a small segment of arc will be:

$$dA = \underbrace{(2\pi y)}_{\text{circumference}} (ds).$$

You may also see S used for surface area (and s used for arc length):

$$dS = (2\pi y)(ds).$$

The surface area of our trumpet shape will then be:

$$\begin{aligned} \text{Surface area} &= \int_0^a \underbrace{2\pi x^2}_{2\pi y} \underbrace{\sqrt{1+4x^2}\,dx}_{ds \text{ from before}} \\ &\vdots \qquad\qquad (\text{substitute } x = \tfrac{1}{2}\tan u) \end{aligned}$$

The calculation for this integral is long; if we wanted to we could use a computer program to get an answer. Our goal is to be able to see that we could get an exact solution to this integral if we had to; i.e. that we could rewrite it as a product of trigonometric functions.