Example: Circular Arc

$$y = \sqrt{1 - x^2}$$

describes the graph of a semicircle. We'll find the arc length of the piece of this semicircle above the interval $0 \le x \le a$. (See Figure 1.)

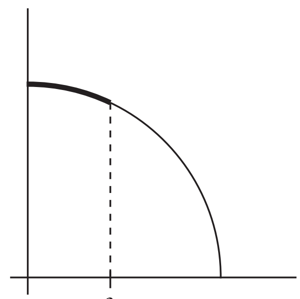

Figure 1: Arc length of $y = \sqrt{1 - x^2}$ over $0 \le x \le a$.

We'll use the variable α to denote the arc length along the circle. We could calculate the exact value of α using trigonometry, but we'll first find it using calculus. We start by finding y'.

$$\begin{aligned} y' &= \frac{-x}{\sqrt{1 - x^2}} \\ ds &= \sqrt{1 + (y')^2}\, dx \\ &= \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\, dx \end{aligned}$$

Yuck. Let's simplify $1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2$ in a separate calculation:

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2 \\ &= 1 + \frac{x^2}{1 - x^2} \\ &= \frac{1 - x^2 + x^2}{1 - x^2} \\ 1 + (y')^2 &= \frac{1}{1 - x^2}. \end{aligned}$$

Plugging this in and using our formula for arc length, we get:

$$\alpha = \int_0^a \frac{dx}{\sqrt{1 - x^2}}$$

$$
\begin{aligned}
&= \sin^{-1} x\Big|_0^a \\
\alpha &= \sin^{-1} a. \\
(\text{So } \sin\alpha &= a.)
\end{aligned}
$$

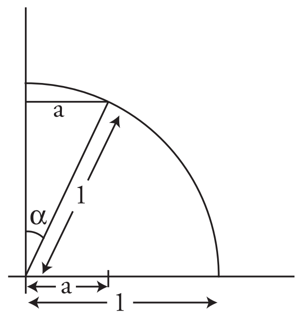

Figure 2: The arc length equals α.

This is a little deeper than it looks; we went a distance α along the arc of a circle that has radius 1 and ended up at a point whose x-coordinate was a.

Previously, if we had an angle whose measure was α radians, we'd say:

$$\sin\alpha = a.$$

You may have been told that radians measured the arc length along the curve of the circle, but this is the first time you've been able to derive it.

Remember that our first definition of the exponential function e^x involved the slope of its graph, but later we were able to define the natural log function as an integral. The same sort of thing is happening here; if you want to know what radians are you have to calculate this arc length. This gives you a new definition of the arcsine function, which gives you a new definition of the sine function, which leads to an improved definition and understanding of trig functions.