Welcome back to recitation. In this video, I'd like us to find an antiderivative of the function 1 over x squared minus 8x plus 1. So I'll give you a while to work on it, and then I'll come back, and I'll show you how I started.

So welcome back. Well, what we'd like to do is, find an antiderivative to 1 over x squared minus 8x plus 1. And how we're going to do that, is we're going to use the technique completing the square. And I'm going to set up the problem, I'm going to get it to a certain place, and then I'm going to let you finish it. And how do you know if you got the right answer? Well, you actually take a derivative of your answer, and see if it gives you back 1 over x squared minus 8x plus 1. That's how you can check.

So let's start off. If I want to complete the square, let's just remind ourselves how to complete the square on this quadratic. So I'd like something right here that makes this a perfect square. Right? Well, the 8 here in the middle, if I want a perfect square, if you think about this, I'm going to have an x minus-- I need a number here that when I multiply it by 2, that's where this 8 comes from, it gives me 8. So obviously I need this number to be a 4. Right? Which puts what here? Puts a 16 here, right?

So just to double check, what I'm looking for is a number here-- I need a number right here that when I multiply by 2, gives me negative 8, and then I need to figure out what it squares to. So that number is negative 4, and it squares to 16. But obviously this isn't what I have, right? I have plus 1. So what have I had to do to get from here to here? Well, I had plus 1 and now I have plus 16, so obviously I've added 15. So I have to subtract 15 to keep this, to keep these three lines all equal to each other.

So to understand where that comes from, let me just remind you, my denominator was looking like this. I'd like it to have a perfect square, and then subtract a constant, or add a constant. Right now I have something that-- I don't have a perfect square in here. I can't make this into a perfect square unless I add a certain amount to the constant right here.

So I had to add a 15 to the constant here. Notice 16 minus 15 is 1. That's my check, also, that the lines are equal. And so what I've done, is I've added 15 and subtracted 15, and then I put this plus 16 into here. x minus 4 quantity squared is exactly these first three terms. And then I keep the minus 15.

Now, you might say, why did you do this? So let's make sure we understand why we're completing the square on this. If we come back, I'm going to put this line in place of what's in the denominator there, because these three things are all equal. So this is actually the integral of dx over x minus 4 quantity squared minus 15.

Now you might say, Christine, this looks no easier. I don't know why you did this. But it actually is one of our favorite, or least favorite, depending on how you feel about it, types of tricks we use now, which is the trig substitution. So some people love this because they just have to memorize a little formula, and some people love it because they can draw a triangle and understand what they choose.

I'm going to show you, remind you what the formula was you saw in class. I believe Professor Jerison said something like this. If the denominator is in the form u squared minus a squared, this implies that you make u equal to a secant theta.

Now, he probably wrote it as x, but I wrote it as u for a very specific reason. Because here I have x minus 4. I have x minus 4 quantity squared. Now, this is where it gets a little rough, right? This is not a perfect square, but it is the perfect square-- it is the square of the square root of 15. So I can write the denominator in the form, something squared minus something else squared.

And again, you might say, why is this good? Well, what we're going to be able to do, is we're going to be able to rewrite this in terms of trigonometric functions, which will make it much simpler to solve.

So let's use what Professor Jerison gave us. And so what we see, is that this is u and this is a. Right? So I get x minus 4 is equal to square root of 15 secant theta.

Now, you might not like this square root of 15, but it's just hanging out. It's not causing any problems. It's just a number there, so we'll keep it a square root of 15. So you don't have to worry about it.

Now what's the point again? Let me just remind you, the object is to get this in terms of the trig functions. So we should anticipate that probably we'll have some tangent functions to go with this. And there are two reasons to think that. The first reason to think that is at some point, I have to find dx. Well, the derivative of secant involves secant and tangent, right? So that's going to pull in a tangent function somewhere.

I'm also going to have a tangent function show up somewhere else. And where that's going to be, is coming from this denominator, this expression in the denominator. Because there's a certain trig identity that we should have memorized, but I'll just remind you. I'll write it here and put a star next to it. It's 1 plus tangent squared theta is equal to secant squared theta. So this is a-- I'll even put a star on the other side. So we should really remember this. Now, where does it come from? It comes from the cosine squared theta plus sine squared theta equals 1 identity. You can divide everything by cosine squared theta and get this one.

So we have this identity, and so if you notice, we're going to be able to manipulate the expression right here, and get the denominator to look like tangent squared theta. So let's do some of that work off to the right here.

So what did I say we needed? We have this expression. We need dx, so let's find-- actually, no. Let's find the denominator first, because I was just talking about it.

So if I look at what x minus 4 squared is, I'm going to substitute in this expression. So x minus 4 squared minus 15 is the same as, based on this substitution, square root 15 secant theta squared minus 15, which is 15 secant squared theta minus 15, which, just to hammer home the point, is 15 times the quantity secant squared theta minus 1.

OK? Everybody follows, hopefully. All I've done is the substitution I made, and then I started expanding, or I squared this term, and I factored out the 15.

And now let's go back to my start expression. What is secant squared theta minus 1? It's tangent squared theta. So we get 15 tangent squared theta.

So that is actually what the denominator of our integral is going to be over there. So I'm going to come in and put that part-- actually, let me even put this here, too. So right now, our denominator is 15 tangent squared theta.

So far, so good. But of course, if I put a dx up here, I'm in trouble. Because I have, it's a function of theta now. So I need to write this-- I shouldn't write in terms of x. I need to figure out what it is in terms of theta. And to do that, we again use the substitution that we made. Which is just above the starred expression. It was that x minus 4 equals square root 15 secant theta. This is going to allow us to find what d theta is in terms of dx. OK? So let's do that.

So I'm not done, by the way, over here. I'm not done I've got a little gap I've got to fill in the numerator.

So let's come back over here. So now we have x minus 4-- let me just write that one more time. So we get dx is equal to the square root of 15. Well, what's the derivative of secant theta? It's secant theta tangent theta d theta.

So now I have all the pieces I need. And I'm actually going to rewrite the whole thing over here underneath, so that I can work with it a little bit more. So the dx is in the numerator. Square root 15 secant theta tan theta d theta, all over 15 tan squared theta.

Now, this might still look a little messy, but we can simplify it some more. We divide out by one tangent, we'll pull this out in front. And notice, what's secant? Secant theta is 1 over cosine theta, and tangent theta is sine theta over cosine theta. So let's write that down. So this becomes square root 15 over 15. We'll just leave it out there. It's not hurting anyone.

So we get a 1 over cosine theta times-- well, tangent theta, 1 over tangent theta is cotangent theta also. There's another way to think about it. So it's cosine theta over sine theta d theta. So these divide out.

And I'm left with, I'm taking now an antiderivative of 1 over sine theta, which is cosecant theta. So I have to find an antiderivative of cosecant theta. Well, you can find that with the exact same strategy you found, or I should say, that Professor Jerison used in class-- or maybe it was actually Professor Miller in that lecture-- to find an antiderivative of secant theta. So you can do the same kind of thing with cosecant theta, because they have the same kind of derivatives. Cosecant and cotangent have very similar-looking derivatives to tangent and secant. Same kinds of relationships. So you can actually find that antiderivative. So this is some constant we don't care about. And once you find that, this will be in terms of theta. Your final answer needs to be in terms of x, but you saw how to do that, actually. You just need to make a triangle that represents the relationship between x and theta. So I'll draw a picture of that triangle, then I'll give a little summary of what we did, and then we'll stop.

So let me draw a picture of that triangle. So from here, all you would do is actually find this antiderivative, and then you would have to make the right kind of substitution in terms of theta. We want to know, how do we find that, do that substitution.

So the triangle is going to come from the following thing. We know x minus 4, again, is square root of 15 secant theta. So I'm going to make this theta. Secant theta, well, it's 1 over cosine theta, cosine is adjacent over hypotenuse, so secant is hypotenuse over adjacent. Right? That's the relationship. So x minus 4 over 15 is equal to the hypotenuse over the adjacent. Did I square root? Sorry. Square root.

So the hypotenuse is x minus 4, the adjacent is square root of 15, and then now I can fill in the opposite by Pythagorean theorem. Right? I just take this squared, and I subtract this squared, and then I take the square root. So I get the square root of x minus 4 squared minus 15.

So whatever I have in terms of theta, I just look at this triangle. If I had in my answer sine theta, I would replace sine theta by this square root divided by x minus 4. Because that's what sine theta is. And so from-- that's how I finish this type of problem, always. I want to have a picture of this triangle, label a theta, use my substitution to give me what two of the sides are, use the Pythagorean theorem to get the third side. So that's the strategy.

So let's go back and just remind ourselves where we came from. We're going to go all the way to the other side. This was a long, long problem.

So what did we do in this problem? I wanted us to find an antiderivative of something. And right away, we can't use partial fractions, because we can't factor out an x here. So I'm forced to use completing the square.

So I completed the square first. That was the little algebra that I had to do first. Then once I have that little bit of algebra, I get into a situation where I'm set up for a trig substitution. So then I had to start off and do some trig substituting. And the things you have to do to make a trig substitution work are, pick the right substitution that makes sense, which you were given that in class. You can also figure it out from a triangle picture, if you wanted to. And then you have to make sure you substitute not just for the expression, the function of x, but also the dx.

So we did all that, and then we came over, further, further, further, further, further, here. And we had everything in terms of theta. So then we had to look at-- we had all these trigonometric functions of theta. We simplified that as far as we could. We got one we could find. Then we finally, we take the antiderivative there, and then in the very end, we're going to substitute in for theta, using the triangle we've drawn up here.

So! I think that's where I'm going to stop this one. Also, I ran out of board space, so I have to stop.