PROFESSOR: So we're going to continue to talk about trig integrals and trig substitutions. This is maybe the most technical part of this course, which maybe is why professor Jerison decided to just take a leave, go AWOL just now and let me take over for him. But I'll do my best to help you learn this technique and it'll be useful for you. So we've talked about trig integrals involving sines and cosines yesterday. There's another whole world out there that involves these other trig polynomials-- trig functions, secant and tangent. Let me just make a little table to remind you what they are. Because I have trouble remembering myself, so I enjoy the opportunity to go back to remind myself of this stuff.

Let's see. The secant is one over one of those things, which one is it? It's weird, it's 1 over the cosine. And the cosecant is 1 over the sin. Of course the tangent, we know. It's the sine over the cosine and the cotangent is the other way around. So when you put a "co" in front of it, it exchanges sine and cosine. Well, I have a few identities involving tangent and secant up there, in that little prepared blackboard up above. Maybe I'll just go through and check that out to make sure that we're all on the same page with them.

So I'm going to claim that there's this trig identity at the top. Secant squared is 1 plus the tangent. So let's just check that out. So the secant is 1 over the cosine, so secant squared is 1 over cosine squared. And then whenever you see a 1 in trigonometry, you'll always have the option of writing as cos^2 + sin^2. And if I do that, then I can divide the cos^2 into that first term. And I get 1 + sin^2 / cos^2. / ^2. Which is the tangent squared. So there you go. That checks the first one. That's the main trig identity that's going to be behind what I talk about today. That's the trigonometry identity part. How about this piece of calculus. Can we calculate what the derivative of tan x is? Actually, I'm going to do that on this board.

So tan x = sin x / cos x. So I think I was with you when we learned about the quotient rule. Computing the derivative of a quotient. And the rule is, you take the numerator and you-- sorry, you take the derivative of the numerator, which is cosine. And you multiply it by the denominator, so that gives you cos^2. And then you take the numerator, take minus the

numerator, and multiply that by the derivative of the denominator, which is -sin x. And you put all that over the square of the denominator. And now I look at that and before my eyes I see the same trig identity, cos^2 + sin^2 = 1, appearing there. This is 1 / cos^2(x) which is secant squared. And, good. So that's what the claim was. The derivative of the tangent is the secant squared.

That immediately gives you an integral. Namely, the integral of secant squared is the tangent. That's the fundamental theorem of calculus. So we verified the first integral there. Well, let's just do the second one as well. So if I want to differentiate the secant, derivative of the secant. So that's d/dx of 1 over the cosine. And again, I have a quotient. This one's a little easier because the numerator's so simple. So I take the derivative of the numerator, which is 0. And then I take the numerator, I take minus the numerator times the derivative of the denominator. Which is -sin x, and put all that over the square of the same denominator. So one minus sign came from the quotient rule, and the other one came because that's the derivative of the cosine. But they cancel, and so I get sin / cos^2, which is sin / cos times 1 / cos and so that's the secant, that's 1 / cos, times tan x.

So, not hard. That verifies that the derivative of secant is secant tangent. And it tells you that the integral of that weird thing in case you ever want to know, the integral of the secant tangent is the secant. Well, there are a couple more integrals that I want to do for you. Where I can't sort of work backwards like that. Let's calculate the integral of the tangent. Just do this straight out. So the tangent is the sine divided by the cosine. And now there's a habit of mind, that I hope you get into. When you see the cosine and you're calculating an integral like this, it's useful to remember what the derivative of the cosine is. Because maybe it shows up somewhere else in the integral. And that happens here. So that suggests we make a substitution. u = cos x. Which means du = -sin x dx. That's the numerator, except for the minus sign.

And so I can rewrite this as, under the substitution, I can rewrite this as -du, that's the numerator, sin x dx is -du, divided by u. Well, I know how to do that integral too. That gives me the natural log, doesn't it. So this is -ln(u) plus a constant. I'm not quite done. I have to back-substitute and replace this new variable that I've made up, called u, with what it is. And what you get is -ln(cos x). So the integral of the tangent is minus log cosine. Now, you find these tables of integrals in the back of the book. Things like that. I'm not sure how much memorization Professor Jerison is going to ask of you, but there is a certain amount of

memorization that goes on in calculus. And this is one of the kinds of things that you probably want to know. Let me do one more integral.

I think I'm making my way through a prepared board here, let's see. Good. So the integral of the tangent is minus log cosine. I'd also like to know what the integral of the secant of x is. And I don't know a way to kind of go straight at this, but let me show you a way to think your way through to it. If I take these two facts, tangent prime is what it is, and secant prime is what it is, and add them together, I get this fact. That the derivative of the sec x + tan x is, well, it's the sum of these two things. Secant squared plus secant tangent. And there's a secant that occurs in both of those terms. So I'll factor it out. And that gives me, I'll put it over here. There's the secant of x that occurs in both terms. And then in one term, there's another secant. And in the other term, there's a tangent.

So that's interesting somehow, because this same term appears on both sides of this equation. Let's write u, for that sec x + tan x. And so the equation that I get is u' = u sec x. I've just made a direct substitution. Just decide that I'm going to write u for that single thing that occurs on both sides of the equation. So u' is on the left, and u * sec x is on the right. Well, there's my secant. That I was trying to integrate. And what it tells you is that = u' / u. Just divide both sides by u, and I get this equation. u' / u, that has a name. Not sure that professor Jerison's used this in this class, but u' / u, we've actually used something like that. It's on the board right now.

It's a logarithmic derivative. It is the derivative of the national logarithm of u. Maybe it's easier to read this from right to left, if I want to calculate the derivative of the logarithm, well, the chain rule says I get the derivative of u times the derivative of the log function, which is 1 / u. So often u' / u is called the logarithmic derivative. But it's done what I wanted. Because it's expressed the secant as a derivative. And I guess I should put in what u is. It's the secant plus the tangent. And so that implies that the integral-- Integrate both sides. That says that the integral of sec x dx, is ln(sec x + tan x). So that's the last line in this little memo that I created. That we can use now for the rest of the class. Any questions about that trick? It's a trick, I have nothing more to say about it.

OK. So, the next thing I-- oh yes, so now I want to make the point that using these rules and some thought, you can now integrate most trigonometric polynomials. Most things that involve powers of sines and cosines and tangents and secants and everything else. For example, let's try to integrate the integral of sec^4 x. Big power of the secant function. Well, there are too

many secants there for me. So let's take some away. And I can take them away by using that trig identity, $\sec^2 = 1 + \tan^2$. So I'm going to replace two of those secants by $1 + \tan^2$. That leaves me with two left over. Now there was method to my madness. Because I've got a secant squared left over there. And secant squared is the derivative of tangent. So that suggests a substitution. Namely, let's say, let's let $u = \tan x$, so that $du = \sec^2 x \, dx$. And I have both terms that occur in my integral sitting there very nicely. So this is the possibility of making this substitution and seeing a secant squared up here as part of the differential here. That's why it was a good idea for me to take two of the secants and write them as $1 + \tan^2$.

So now I can continue this. Under that substitution, I get 1. Oh yeah, and I should add the other fact, that-- Well I guess it's obvious that tangent squared is u^2. So I get $1 + u^2$. And then du-- $\sec^2 \sec^2 x \, dx$, that is du. Well that's pretty easy to integrate. So I get $u + u^3 / 3$. Plus a constant. And then I just have to back-substitute. Put things back in terms of the original variables. And that gives me tan x plus tangent cubed over 3. And there's the answer. So we could spend a lot more time doing more examples of this kind of polynomial trig thing. It's probably best for you to do some practice on your own. Because I want to talk about other things, also. And what I want to talk about is the use of these trig identities in making really trig substitution integration.

So we did a little bit of this yesterday, and I'll show you some more examples today. Let's start with a pretty hard example right off the bat. So this is going to be the integral of dx over x^2 times the square root of $1+x^2$. It's a pretty bad looking integral. So how can we approach this? Well, the square root is the ugliest part of the integral, I think. What we should try to do is write this square root in some nicer way. That is, figure out a way to write $1+x^2$ as a square. That'll get rid of the square root. So there is an example of a way to write 1 plus something squared in a different way. And it's right up there. $\sec^2 = 1 + \tan^2$. So I want to use that idea. And when I see this form, that suggests that we make a trig substitution and write x as the tangent of some new variable. Which you might as well call theta, to because it's like an angle. Then $1+x^2$ is the secant squared. According to that trig identity. And so the square root of $1+x^2$ is sec(theta). Right? So this identity is the reason that the substitution is going to help us. Because it gets rid of the square root and replaces it by some other trig function.

I'd better be able to get rid of the dx, too. That's part of the substitution process. But we can do that, because I know what the derivative of the tangent is. It's secant squared. So dx / d theta is $\sec^2(\theta)$. So dx is $\sec^2(\theta) \, d\theta$. So let's just substitute all of that stuff in, and

rewrite the entire integral in terms of our new variable, theta. So dx is in the numerator. That's sec^2(theta) d theta. And then the denominator, well, it has an x^2. That's tan^2(theta). And then there's this square root. And we know what that is in terms of theta. It's sec(theta).

OK, now. we've done the trig substitution. I've gotten rid of the square root, I've got everything in terms of trig functions of the new variable. Pretty complicated trig function. This often happens, you wind up with a complete scattering of different trig functions in the numerator and denominator and everything. A systematic thing to do here is to put everything in terms of sines and cosines. Unless you can see right away, how it's going to simplify, the systematic thing to do is to rewrite in terms of sines and cosines. So let's do that. So let's see. The secant squared, secant is 1 over cosine. So I'm going to put a cosine squared in the denominator. Oh, I guess the first thing I can do is cancel. Let's do that. That's clever. You were all thinking that too. Cancel those. So now I just get one cosine denominator from the secant there in the numerator. It's still pretty complicated, secant over tangent squared, who knows. Well, we'll find out. Because the tangent is sine over cosine. So I should put a sine squared where the tangent was, and a cosine squared up there. And I still have d theta. And now you see some more cancellation occurs. That's the virtue of writing things out in this way. So now, the square here cancels with this cosine. And I'm left with cos(theta) d theta / sin^2(theta). That's a little simpler.

And it puts me in a position to use the same idea I just used. I see the sine here. I might look around in this integral to see if its derivative occurs anywhere. The differential of the sine is the cosine. And so I'm very much inclined to make another substitution. Say, u, direct substitution this time. And say u is the cosine of theta. Because then du-- Oh, I'm sorry. Say, u is the sine of theta. Because then du is cos(theta) d theta. And then this integral becomes, well, the numerator just is du. The denominator is u^2. And I think we can break out the champagne, because we can integrate that one. Finally get rid of the integral sign. Yes sir.

STUDENT: [INAUDIBLE]

PROFESSOR: OK, how do I know to make u equal to sine rather than cosine. Because I want to see du appear up here. If I'd had a sine up here, that would be a signal to me that maybe I should say let u be the cosine. OK? Also, because this thing in the denominator is something I want to get rid of. It's in the denominator. So I'll get rid of it by wishful thinking and just call it something else. It works pretty well in this case. Wishful thinking doesn't always work so well. So I integrate u^(-2) du, and I get - -1/u plus a constant, and I'm done with the calculus part of this

problem. I've done the integral now. Gotten rid of the integral sign. But I'm not quite done with the problem yet, because I have to work my way back through two substitutions. First, this one. And then this one. So this first substitution isn't so bad to get rid of, to undo, to back-substitute. Because u is just sin(theta). And so 1/u is, I guess a fancy way to write it is the cosecant of theta. 1 over the sine is the cosecant. So I get -csc(theta) plus a constant. Is there a question in the back? Yes sir?

STUDENT: [INAUDIBLE]

PROFESSOR: I'm sorry, my hearing is so bad.

STUDENT: [INAUDIBLE]

PROFESSOR: How did I know this substitution in the first place. It's because of the 1 + x^2. And I want to make use of the trig identity in the upper left-hand corner. I'll make you a table in a few minutes that will put all this in a bigger context. And I think it'll help you then. OK, I'll promise. So, what I want to try to talk about right now is how to rewrite a term like this. A trig term like this, back in terms of x. So I want to undo this trick substitution. This is a trig sub. And what I want to do now is try to undo that trig sub. And I'll show you a general method for undoing trig substitutions. This happens quite often. I don't know what the cosecant of theta is. But I do know what the tangent of theta is. So I want to make a relation between them. OK, so undoing. Trig subs.

So let's go back to where trigonometry always comes from, this right angled triangle. The theta in the corner, and then these three sides. This one's called the hypotenuse. This one is called the adjacent side, and that one's called the opposite side. And now, let's find out where x lies in this triangle. Let's try to write the sides of this triangle in terms of x. And what I know is, x is the tangent of theta. So the tangent of theta, tangent of this angle, is opposite divided by adjacent. Did you learn SOH CAH TOA? OK, so it's opposite divided by adjacent. Is the tangent. So there are different ways to do that, but why not just do it in the simplest way and suppose that the adjacent is 1, and the opposite is x.

This is correct now, isn't it? I get the correct value for the tangent of theta by saying that the lengths of those are 1 and x. And that means that the hypotenuse has length 1 + x^2. Well, here's a triangle. I'm interested in computing the cosecant of theta. Where's that appear in the triangle? Well, let's see. The cosecant of theta is 1 over the sine. And the sine is opposite over hypotenuse. So the cosecant is hypotenuse over opposite. And the hypotenuse is the square

root of 1 + x^2, and the opposite is x. And so I've done it. I've undone the trig substitution. I've figured out what this cosecant of theta is, in terms of x. And so the final answer is minus the square root of 1+x^2, over x, plus a constant, and there's an answer to the original problem.

This took two boards to go through this. I illustrated several things. Actually, this three half boards. I illustrated this use of trig substitution, and I'll come back to that in a second. I illustrated patience. I illustrated rewriting things in terms of sines and cosines, and then making a direct substitution to evaluate an integral like this. And then there's this undoing all of those substitutions. And it culminated with undoing the trig sub. So let's play a game here. Why don't we play the game where you give me-- So, there's a step in here that I should have done. I should've said this is -cos(arctan(theta)) plus a constant. The most straightforward thing you can do is to say since x is the tangent of theta, that means that-- sorry, if x, that means that theta is the arctangent of x. And so let's just put in theta as the arctangent of x, and that's what you get. So really, what I just did for you was to show you a way to compute some trig function applied to the inverse of another trig function. I computed cosecant of the arctangent by this trick.

So now, let's play the game where you give me a trig function and an inverse trig function, and I try to compute what the composite is. OK. So who can give me a trig function. Has to be one of these standard ones.

STUDENT: Tan.

PROFESSOR: Tangent. Alright. How about another one?

STUDENT: Sine.

PROFESSOR: Sine. Do we have agreement on sine.

STUDENT: [INAUDIBLE]

PROFESSOR: Secant?

STUDENT: [INAUDIBLE]

PROFESSOR: Right, csc has the best cheer. So that's the game. We have to compute, try to compute, that composite. Something wrong with this?

STUDENT: [INAUDIBLE]

PROFESSOR: What does acceptable mean? Don't you think-- so the question is, isn't this a perfectly acceptable final answer. It's a correct final answer. But this is much more insightful. And after all the original thing was involving square roots and things, this is the kind of thing you might hope for is an answer. This is just a nicer answer for sure. And likely to be more useful to you when you go on and use that answer for something else. OK, so let's try to do this this. Undo a trig substitution that involved a cosecant. And I manipulate around, and I find myself trying to find out what's the tangent of theta. So here's how we go about it. I draw this triangle. Theta is the angle here. This is the adjacent, opposite, hypotenuse. So, the first thing is how can I make the cosecant appear here, csc x. What dimensions should I give to the sides in order for the cosecant of x, sorry, in order for theta to be the cosecant of x. This thing is theta. So, that means that the cosecant of x-- that means the cosecant of theta should be x. Theta is the arccosecant, so x is the cosecant of theta. So, what'll I take the sides to be, to get the cosecant? The cosecant is 1 over the sine. And the sine is the opposite over the hypotenuse. So I get hypotenuse over opposite. And that's supposed to be what x is. So I could make the opposite anything I want, but the simplest thing is to make it 1. Let's do that. And then what does that mean about the rest of the sides? Hypotenuse had better be x. And then I've recovered this. So here's a triangle that exhibits the correct angle. This remaining side is going to be useful to us. And it is the square root of x^2 - 1.

So I've got a triangle of the correct angle theta, and now I want to compute the tangent of that angle. Well, that's easy. That's opposite divided by adjacent. So I get 1 over the square root of x^2 - 1. Very flexible tool that'll be useful to you in many different times. Whenever you have to undo a trig substitution, this is likely to be useful.

OK, that was a good game. No winners in this game. We're all winners. No losers, we're all winners. OK. So, good. So let me make this table of the different trig substitutions, and how they can be useful. Summary of trig substitutions. So over here, we have, if you see, so if your integrand contains, make a substitution to get. So if your integrand contains, I'll write these things out as square roots. If it contains the square root a^2 - x^2, this is what we talked about on Thursday. When I was trying to find the area of that piece of a circle. There, I suggested that we should make the substitution x = a cos(theta). Or, x = a sin(theta). Either one works just as well. And there's no way to prefer one over the other. And when you make the substitution, x = a cos(theta), you get a^2 - a^2 cos^2(theta). theta. 1 - cos^2 is sin^2. So you get a sin(theta). So this expression becomes equal to this expression under that substitution.

And then you go on. Then you've gotten rid of the square root, and you've got a trigonometric integral that you have to try to do. If you made the substitution a sin(theta), you'd get a a^2 - a^2 sin^2, which is a cos(theta). And then you can go ahead as well. We just saw another example. Namely, if you have a^2 + x^2. That's like the example we had up here. a = 1 in this example. What did we do? We tried the substitution x = a tan(theta). And the reason is that I can plug into the trig identity up here in the upper left. And replace a^2 + x^2 by a sec(theta). Square root of the secant squared. There's one more thing in this table. Sort of, the only remaining sum or difference of terms like this. And that's what happens if you have x^2 - a^2. So there, I think we can make a substitution a sec(theta). Because, after all, sec^2(theta)-- so x^2 - a^2-- Sorry. Let's see what happens when I make that substitution. x^2 - a^2 = a^2 sec^2(theta) - a^2. Under this substitution. That's sec^2 - 1. Well, put the 1 on the other side. And you find tan^2, coming out. So this is a a^2 tan^2(theta). And so that's what you get, a times tan(theta). After I take the square root, I get a tan(theta). So these are the three basic trig substitution forms. Where trig substitutions are useful to get rid of expressions like this, and replace them by trigonometric expressions. And then you use this trick, you do the integral if you can and then you use this trick to get rid of the theta at the end.

So now, the last thing I want to talk about today is called completing the square. And that comes in because unfortunately, not every square root of a quadratic has such a simple form. You will often encounter things that are not just the square root of something simple. Like one of these forms. Like there might be a middle term in there. I don't actually have time to show you an example of how this comes out in a sort of practical example. But it does happen quite frequently. And so I want to show you how to deal with things like the following example. Let's try to integrate dx over x^2 + 4x, the square root of x^2 + 4x.

So there's a square root of some square, some quadratic. It's very much like this business. But it isn't of any of these forms. And so what I want to do is show you how to rewrite it in one of those forms using substitution, again. All this is about substitution. So the game is to rewrite quadratic as something like x plus something or other. Plus some other constant. So write it, try to write it, in the form of a square plus or minus another constant. And then we'll go on from there. So let's do that in this case. x^2, x^2 + 4x. Well, if you square this form out, then the middle term is going to be 2ax. So that, since I have a middle term here, I pretty much know what a has to be. The only choice in order to get something like x^2 + 4x out of this, is to take a to be 2. Because then, this is what you get. This isn't quite right yet, but let's compute what I have here. x^2 + 4x, so far so good, plus 4, and I don't have a plus 4 here. So I have to fix that

by subtracting 4. So that's what I mean. I've completed the square. The word for this process of eliminating the middle term by using the square of an expression like that. That's called completing the square.

And we can use that process to compute this integral. So let's do that. So I can rewrite this integral, rewrite this denominator like this. And then I'm going to try to use one of these forms over here. So in order to get a single variable there, instead of something complicated like x + 2, I'm inclined to come up with another variable name and write it equal, write x + 2 as that other variable name. So here's another little direct substitution. u = x + 2. Figure out what du is. That's pretty easy. And then rewrite the integral in those terms. So dx = du. And then in the denominator I have, well, I have the square root of that. Oh yeah, so I think as part of this I'll write out what x^2 + 4x is. The point is, it's equal to u^2 - 4. x^2 + 4x = u^2 - 4.

There's the data box containing the substitution data. And so now I can put that in. I have x^2 + 4x there. In terms of u, that's u^2 - 4. Well, now I'm in a happier position because I can look for u^2 - 4 for something like that in my table here. And it actually sits down here. So except for the use of the letter x here instead of u over there. That tells me what I want. So to handle this, what I should use is a trig substitution. And the trig substitution that's suggested is, according to the bottom line with a = 2, so a^2 = 4. The suggestion is, I should take x-- But I'd better not use the letter x any more. But I don't have a letter x, I have the letter u. I should take u equal to 2 secant. And then some letter I haven't used before. And theta is available. This is a look-up table process. I see the square root of u^2 - 4, I see that that's of this form. I'm instructed to make this substitution. And that's what I just did. Let's see how it works out.

So that means the du is 2, OK. What's the derivative of the secant? Secant tangent. So du = 2 sec(theta) tan(theta). And u^2 - 4 is, here's the payoff. I'm supposed to be able to rewrite that in terms of the tangent. According to this. u u^2 - 4 is 4 secant squared minus 4. And secant squared minus 1 is tangent squared. So this is 4 tan^2(theta). Right, yeah?

STUDENT: [INAUDIBLE]

PROFESSOR: But I squared it. And now I'll square root it. And I'll get a 2 and this tangent will go away. So there's my data box for this substitution. And let's go on to another board. So where I'm at is the integral of du over the square root of u^2 - 4. And I have all the data I need here to rewrite that in terms of theta. So du = 2 sec(theta) tan(theta) d theta. And the denominator is 2 tan(theta). Ha. Well, so some very nice simplification happens here. The 2's cancel. And the

tangents cancel. And I'm left with trying to work with the integral, sec(theta) d theta. And luckily enough at the very beginning of the hour, I worked out how to compute the integral of the secant of theta. And there it is. So this is ln(sec(theta) + tan(theta)) plus a constant.

And we're done with the calculus part. There's no more integral there. But I still am not quite done with the problem, because again I have these two substitutions to try to undo. So let's undo them one by one. Let's see. I have this trig substitution here. And I could use my triangle trick, if I need to. But maybe I don't need to. Let's see, do I know what the secant of theta is in terms of u? Well, I do. So I get ln(u/2). Do I know what the tangent is in terms of u? Well, I do. It's here. So I lucked out, in this case. And I don't have to go through and use that triangle trick.

So the tangent of theta is the square root of u^2 - 4, over 2. Good. So I've undone this trig substitution. I'm not quite done yet because my answer is involved with u. And what I wanted originally was x. But this direct substitution that I started with is really easy to deal with. I can just put x + 2 every time I see a u. So this is the natural logarithm of (x+2)/2 plus the square root... What's going to happen when I put x + 2 in place for u here? You know what you get. You get exactly what we started with. Right? I put x + 2 in place of the u here. I get x^2 + 4x. So I've gotten back to a function purely in terms of x OK, that's a good place to quit. Have a great little one-day break. I guess this class doesn't meet on Monday anyway. Bye.