The Area Under the Bell Curve

Our last example of a definite integral is:

$$F(x) = \int_0^x e^{-t^2}\, dt.$$

As we've already remarked, this is a new function that we can't express in terms of functions we already know. To begin to understand the properties of this function F we'll sketch its graph.
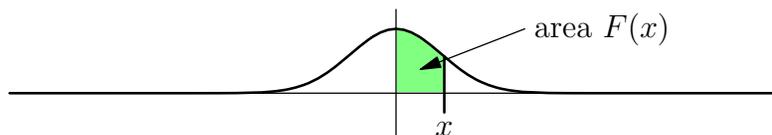

Figure 1: Area under e^{-t^2}.

The fundamental theorem tells us right away that:

$$F'(x) = e^{-x^2}.$$

It's easy to find the value of $F(0)$, because the "starting place" of the integral is 0.

$$F(0) = \int_0^0 e^{-t^2}\, dt = 0.$$

We can also compute the second derivative:

$$F''(x) = -2xe^{-x^2}$$

The first derivative is always positive, so $F(x)$ is always increasing. Because the sign of $-2xe^{-x^2}$ is just the sign of $-2x$, its graph will be concave down when $x > 0$ and concave up when $x < 0$. And finally, $F'(0) = e^{-0^2} = 1$. (We often define functions so that their graphs have slope 1 in convenient locations.)

By combining all this information we get a pretty good idea of what the graph of the function looks like, even though we cannot write down a strictly algebraic equation for $F(x)$.

We want to know as much as possible about this function, so we'll discuss a few more features before moving on.

First we show that $F(x)$ is odd. As mentioned in Figure 1, $F(x)$ is the area under the graph of e^{-t^2} between 0 and x. From the symmetry of the graph we see:

$$\int_{-x}^0 e^{-x^2}\, dx = \int_0^x e^{-x^2}\, dx,$$

so:

$$F(-x) = \int_0^{-x} e^{-x^2}\, dx = -\int_{-x}^0 e^{-x^2}\, dx = -\int_0^x e^{-x^2}\, dx = -F(x).$$
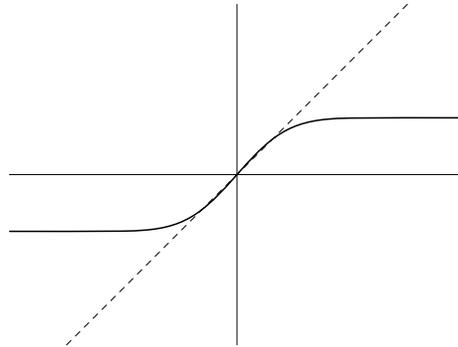
Figure 2: Graph of $F(x) = \int_0^x e^{-t^2}\, dt$.

We conclude that $F(x)$ is an odd function.

Because F is odd we know that the part of its graph to the right of the y-axis is exactly a rotation of the part of its graph to the left of the y-axis. If we know the shape of one branch we immediately know the shape of the other branch.

Our final step to understanding the graph is to figure out what happens at the ends. It turns out that the graph approaches a horizontal asymptote as x approaches positive infinite and, by symmetry, as x goes to negative infinity. On the right, the graph rises to a certain level; on the left it falls to the negative of that level.

What is that level? It's the area under the graph of $f(x) = e^{-t^2}$ between 0 and infinity: $\frac{\sqrt{\pi}}{2}$. It took several years for mathematicians to discover the exact value of this area.

$$\lim_{x \to \infty} F(x) = \frac{\sqrt{\pi}}{2}$$

$$\lim_{x \to -\infty} F(x) = -\frac{\sqrt{\pi}}{2}$$

Because 1 is a nicer number to work with than $\frac{\sqrt{\pi}}{2}$, people define the *error function* to be:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} F(x).$$

This is a famous function, as is its cousin the standard normal distribution.

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