Slope of a line tangent to a circle – direct version

A circle of radius 1 centered at the origin consists of all points (x, y) for which $x^2 + y^2 = 1$. This equation does not describe a function of x (i.e. it cannot be written in the form $y = f(x)$). Indeed, any vertical line drawn through the interior of the circle meets the circle in two points — every x has two corresponding y values. Let's see what goes wrong if we attempt to solve the equation of a circle for y in terms of x.

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + y^2 - x^2 &= 1 - x^2 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1 - x^2} \end{aligned}$$

This still isn't a function because we get two choices for y — positive or negative. However, we do get a function if we look just at the positive case (i.e. at just the top half of the circle), and we can then find $\frac{dy}{dx}$, which will be the slope of a line tangent to the top half of the circle.

To compute this derivative, we first convert the square root into a fractional exponent so that we can use the rule from the previous example.

$$y = \sqrt{1 - x^2} = (1 - x^2)^{\frac{1}{2}}$$

Next, we need to use the chain rule to differentiate $y = (1 - x^2)^{\frac{1}{2}}$. The outside function is $u^{1/2}$ and the inside function is $1 - x^2$, so the chain rule tells us that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{2}u^{-1/2} \cdot (-2x) = -x \cdot (1 - x^2)^{-1/2} = \frac{-x}{\sqrt{1 - x^2}}.$$

If we want, we can use the fact that $y = \sqrt{1 - x^2}$ to rewrite this as $y' = -x/y$.

We conclude that the slope of the line tangent to a point (x, y) on the top half of the unit circle is $-x/y$.