Exam 1 Review, Continued

The Definition of the Derivative

The main thing we talked about in the first part of the course was the definition of the derivative; one of our goals has been to understand its meaning. The formula for the derivative is:

$$\frac{d}{dx}f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This is a central focus of this course and you want to be able to recognize this formula in a number of forms.

How might you be asked to use this on a test? Our chief use of this formula occurred while finding specific formulas for derivatives. In fact, we used it to find many of the formulas you might be tested on:

$$x^n,\ \sin^{-1} x,\ \tan^{-1} x,\ \sin x,\ \cos x,\ \tan x,\ \sec x,\ e^x,\ \ln x.$$

Let's briefly review which functions we used this on, and what other facts we needed in finding our formulas. We used the definition of the derivative to compute formulas for the derivatives of

$$1/x,\ x^n,\ \sin x,\ \cos x,\ a^x,\ u \cdot v \text{ and } u/v.$$

To complete these calculations we needed to know the derivatives of $\sin x$, $\cos x$ and a^x at $x = 0$. To derive the product and quotient rules we needed to know the slopes of the individual functions u and v. It wasn't the definition of the derivative alone that got us these formulas, but in each case it got us to something simpler that we could use to get our formulas. That "getting to something simpler" is the sort of thing you should be able to do on a test.

But don't sit down and memorize the derivations of all those formulas to prepare for the test. What you need to learn is how to use the definition of the derivative to derive a formula for any function — there could be questions on e^x, $\frac{1}{x^2}$, or some other function.

Limits

There are also some fundamental limits you should know about for the test.

Recall that any equation can be read in two directions. In the case of the definition of the derivative, reading the equation backward tells us that if we know the slope of the graph of a function you can find the value of a limit:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x).$$

For example, suppose you are asked to find:

$$\lim_{u \to 0} \frac{e^u - 1}{u}.$$

If you recognize that:

$$\lim_{u\to 0} \frac{e^u - 1}{u} = \left.\frac{d}{du}e^u\right|_{u=0}$$

then it's easy to see that the answer is 1. The key is recognizing that

$$\lim_{u\to 0} \frac{e^u - 1}{u}$$

matches the formula

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

when $f(x) = e^x$, $\Delta x = u$, and $x = 0$.

Implicit Differentiation (continued)

For the test, you also should be able to use implicit differentiation to derive formulas for derivatives of functions like $\sin^{-1} x$ and $\ln x$. The power of implicit differentiation is that it lets you take a formula and simplify it as much as possible; you're not restricted to writing y as a function of x.

For instance, $\sin y = x$ is a much simpler equation than $y = \sin^{-1} x$. This second formula is easy to differentiate implicitly:

$$(\cos y)y' = 1$$

so

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

(If you're not sure why $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, draw a right triangle whose "opposite" edge has length x and whose hypotenuse has length 1.)

Tangent Lines

Geometrically, a derivative is the slope of a tangent line to a graph. On the test you may be asked to compute the equation of a tangent line, to graph the function y' or to tell whether a function is differentiable by looking at its graph.

The first two types of questions are fairly straightforward, and we really only have one way to answer the third type of question. If a graph is differentiable, it must be possible to draw a line tangent to every point on the graph. This means that the limit of the secant lines must be well defined at every point on the graph. And this, in turn, means that the secant lines must have the same limits as they approach the tangent from the left and from the right.

One way to practice for the test is to graph a function like $y = \ln x$ and then try to draw the graph of y'. To do this, first draw a few tangent lines to the graph. Observe what you can about them – are their slopes all positive? All increasing? All the same? Do any tangent lines have slope zero? How steep is the steepest tangent line, or is there no steepest line? Then use that information to graph a curve whose height above the x-axis (approximately) equals the slopes of the tangents to the original graph.

Do not expect the graph of the derivative to look like the graph of the function. If possible, check your work by taking the derivative of the function whose graph you looked at.