<u>The fundamental theorems of calculus.</u>

Here are the two basic theorems relating integrals and derivatives. You should know the proofs of these theorems.

First, we need to discuss "one-sided" derivatives.

If a function is defined on an interval [a,b], we know what it means for f to be <u>continuous</u> on $[a,b]$. It means that f is continuous in the ordinary sense at each point of the open interval (a,b), and that f satisfies the appropriate version of one-sided continuity at each of the end points a and b.

What shall it mean for f to be <u>differentiable</u> on $[a,b]$? It will mean that f is differentiable in the ordinary sense at each point of (a,b), and that the appropriate one-sided derivatives of f exist at the end points. More specifically, the one-sided derivative of f at a is the one-sided limit

$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} .$$

Similarly, the one-sided derivative of f at b is the one-sided limit

$$f'(b) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h} .$$

Of course, if it happens that f is defined and differentiable in some open interval that contains $[a,b]$, then it is automatically true that f is differentiable on $[a,b]$, in the sense just defined. This is the situation that usually occurs in practice.

Now we prove a lemma:

Lemma 1. Suppose f *is integrable on the closed interval having* c *and* d *as end points and that* $|f(x)| \le M$ *on this interval. Then*

$$\left|\int_c^d f\right| \le M|d - c|.$$

Proof. Assume first that $c < d$. Now

$$-M \le f(x) \le M$$

for all x in $[c,d]$. The comparison theorem for integrals tells us that

$$-M(d-c) \le \int_c^d f \le M(d-c).$$

On the other hand, if $d < c$, the comparison theorem tells us that

$$-M(c-d) \le \int_d^c f \le M(c-d).$$

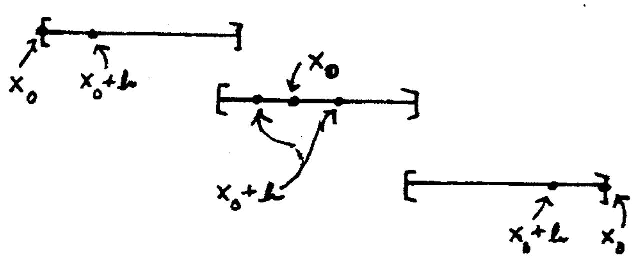
In either case, we conclude that $\left|\int_c^d f\right| \le M|d-c|$. □

Theorem 2. *Suppose* f *is integrable on* $[a,b]$. *Let* c *be a point of* $[a,b]$. *Let*

$$A(x) = \int_c^x f(t)\,dt$$

for x *in* $[a,b]$. *Then* $A(x)$ *is continuous on* $[a,b]$.

Proof. Throughout this proof, let h denote a number such that $h \neq 0$ and $x_0 + h$ is in $[a,b]$. This means that



h is small, and that h is positive if $x_0 = a$, and h is negative if $x_0 = b$.

We know f is bounded on $[a,b]$; choose M so that $|f(x)| \leq M$ for x in $[a,b]$. Then we compute

$$A(x_0+h) - A(x_0) = \int_c^{x_0+h} f - \int_c^{x_0} f$$

$$= \int_{x_0}^{x_0+h} f(x)\,dx.$$

By the preceding lemma, we have

$$|A(x_0+h)-A(x_0)| = \left|\int_{x_0}^{x_0+h} f(x)\,dx\right| \leq M|h|.$$

We use this inequality to show that $A(x)$ is continuous at x_0. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then if $|h| < \delta$, the above inequality shows that

$$|A(x_0+h) - A(x_0)| \leq M|h| < M(\varepsilon/M) = \varepsilon. \quad \square$$

Theorem 3. (First fundamental theorem of calculus.) *Let* f *be integrable on* $[a,b]$; *let* c *be a point of* $[a,b]$. *Let*

$$A(x) = \int_c^x f(t)\, dt.$$

If f *is continuous at the point* x_0 *of* $[a,b]$, *then* $A'(x_0)$ *exists and* $A'(x_0) = f(x_0)$.

Proof. Let h be as in the preceding proof. As before, we compute

$$A(x_0+h) - A(x_0) = \int_{x_0}^{x_0+h} f(t)\, dt.$$

Now since $f(x_0)$ is a constant, we have the equation

$$f(x_0)\cdot h = \int_{x_0}^{x_0+h} f(x_0)\, dt.$$

Subtracting and using linearity, we see that

$$(*) \qquad \frac{A(x_0+h) - A(x_0)}{h} - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} (f(t)-f(x_0))dt.$$

To prove that $A'(x_0)$ exists and equals $f(x_0)$ is equivalent to showing that

$$\lim_{h \to 0} \frac{A(x_0+h) - A(x_0)}{h} = f(x_0).$$

(The limit is a one-sided limit if x_0 equals a or b). To prove this statement, it suffices to show that the right side of (*) approaches zero.

We use the continuity of f at x_o. Given $\varepsilon > 0$, choose $\delta > 0$ so that

$$|f(x) - f(x_o)| < \varepsilon$$

whenever $|x-x_0| < \delta$ and x is in $[a,b]$. Then if $0 < |h| < \delta$, the inequality

$$|f(x) - f(x_0)| < \varepsilon$$

holds for all x in the interval having end points x_0 and $x_0 + h$. It follows from the preceding lemma that

$$\left| \int_{x_0}^{x_0+h} (f(x) - f(x_0))dx \right| \leqslant \varepsilon |h|.$$

We conclude that for $0 < |h| < \delta$,

$$\left| \frac{A(x_0+h) - A(x_0)}{h} - f(x_0) \right| \leqslant \varepsilon,$$

as desired. □

Theorem 4. (Second fundamental theorem of calculus.) Suppose $P(x)$ is defined on $[a,b]$ and that $P'(x)$ exists and is continuous on $[a,b]$. Let c be a point of $[a,b]$. Then for all x in $[a,b]$,

$$\int_c^x P'(t)dt = P(x) - P(c).$$

Proof. Since $P'(x)$ is continuous on $[a,b]$, it is integrable. Furthermore, if

$$A(x) = \int_c^x P',$$

then by the first fundamental theorem, $A'(x)$ exists and equals $P'(x)$. We conclude that the function $P(x) - A(x)$ is continuous on $[a,b]$ (in fact, differentiable on $[a,b]$) and that its derivative vanishes on $[a,b]$.

It follows from the mean-value theorem (see p. 187 of the text) that $P(x) - A(x)$ is constant on $[a,b]$. Let

$$P(x) - A(x) = K$$

for all x in $[a,b]$. Setting $x = c$, we see that

$$P(c) - 0 = K.$$

Therefore,

$$A(x) = P(x) - K = P(x) - P(c),$$

Definition. If $f(x)$ is a function defined on $[a,b]$, a primitive of f is a function $P(x)$ defined on $[a,b]$ such that $P'(x) = f(x)$. (Such a function P does not always exist, of course.) We also call $P(x)$ an antiderivative of f, and we write

$$\int f(x)\, dx = P(x) + C.$$

The second fundamental theorem says that if f is continuous, one can compute $\int_a^b f$ provided one can find a primitive P of f; for then $\int_a^b f = P(b) - P(a)$.

Remark. These two theorems may be summarized as follows:

(1) $$\frac{d}{dx}\int_c^x f = f(x) \quad \underline{\text{if}}\ f\ \underline{\text{is continuous at}}\ x.$$

(2) $$\int_c^x \frac{d}{dx} P = P(x) - P(c) \quad \underline{\text{if}}\ \frac{dP}{dx}\ \underline{\text{is continuous on the interval having end points}}\ c\ \underline{\text{and}}\ x.$$

These theorems say, in essence, that integration and differentiation are inverse operations. But in each case, there is a continuity requirement that the integrand must satisfy in order for the theorem to hold.

Corollary 5. Let r be a rational constant with $r \neq -1$. If a and b are positive real numbers, then

$$\int_a^b x^r\, dx = \frac{b^{r+1} - a^{r+1}}{r+1}.$$

Proof. Let $P(x) = x^{r+1}/(r+1)$ for all $x > 0$. Then we have shown (see notes I) that $P'(x) = x^r$ for all $x > 0$. Since the function x^r is continuous for all $x > 0$, it is continuous on $[a,b]$, so the second fundamental theorem applies to give our formula. □

Exercises

1. If $b > 0$, show that

$$\int_0^b [t]\, dt = \tfrac{1}{2}[b]\,(2b - [b] - 1).$$

[Hint: Let $n = [b]$. Evaluate $\int_0^n [t]\, dt$ and $\int_n^b [t]\, dt$.]

2. Let $A(x) = \int_0^x [t]\, dt$.

(a) Use the first fundamental theorem of calculus to show that $A'(x) = [x]$ when x is not an integer, and that $A'(x)$ does not exist when x is an integer. See the figure on p. 127 of Apostol.

(b) Use the formula of Exercise 1 to verify the same result.

3. Use the chain rule to evaluate:

(a) $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \frac{dt}{1+t^5}$.   (b) $\dfrac{d}{dx}\displaystyle\int_{x^3}^{1} \frac{dt}{1+t^5}$.   (c) $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^2} \frac{dt}{1+t^5}$.

4. Suppose $F(t)$ is continuous for $a \leq t \leq b$. Let

$$A(x) = \int_a^x F(t)\, dt$$

for x in $[a,b]$.

(a) Suppose $g(u)$ is a function whose values lie in the interval $[a,b]$, with g differentiable. Consider the function

$$B(u) = A(g(u)) = \int_a^{g(u)} F(t)\,dt.$$

Use the chain rule to show that

$$B'(u) = F(g(u))g'(u).$$

We express this fact in words as follows: The derivative of

$$\int_a^{g(u)} F(t)\,dt$$

with respect to u equals the integrand, evaluated at the upper limit, times the derivative of the upper limit.

(b) If $g(u)$ and $h(u)$ are two functions whose values lie in $[a,b]$, and if g and h are differentiable, derive a formula for the derivative with respect to u of

$$\int_{h(u)}^{g(u)} F(t)\,dt.$$

[Hint: Write

$$\int_h^g F = \int_a^g F - \int_a^h F.\ ]$$

5. Suppose f is integrable on $[a,b]$. Let

$$A(x) = \int_a^x f(t)\,dt$$

for x in $[a,b]$. Let x_0 be a point of (a,b).

(a) If f is continuous at x_0, what can you say about the function $A(x)$?

(b) If f is continuous on $[a,b]$, what can you say about $A(x)$?

(c) If f is continuous from the right at x_0, what can you say about $A(x)$? [Hint: Examine the proof of the first fundamental theorem.]

(d) If f' exists on $[a,b]$ what can you say about $A(x)$?

Justify your answers, using the theorems we have proved.