Lecture 5. September 16, 2005

Homework. Problem Set 2 Part I: (a)–(e); Part II: Problem 2.

Practice Problems. Course Reader: 1I-1, 1I-4, 1I-5

1. Example of implicit differentiation. Let $y = f(x)$ be the unique function satisfying the equation,

$$\frac{1}{x} + \frac{1}{y} = 2.$$

What is slope of the tangent line to the graph of $y = f(x)$ at the point $(x, y) = (1, 1)$?

Implicitly differentiate each side of the equation to get,

$$\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d(2)}{dx} = 0.$$

Of course $(1/x)' = (x^{-1})' = -x^{-2}$. And by the rule $d(u^n)/dx = nu^{n-1}(du/dx)$, the derivative of $1/y$ is $-y^{-2}(dy/dx)$. Thus,

$$-x^{-2} - y^{-2}\frac{dy}{dx} = 0.$$

Plugging in x equals 1 and y equals 1 gives,

$$-1 - 1y'(1) = 0,$$

whose solution is,

$$y'(1) = \boxed{-1}.$$

In fact, using that $1/y$ equals $2 - 1/x$, this can be solved for every x,

$$\frac{dy}{dx} = (x^{-2})/(y^{-2}) = \frac{1}{x^2} \cdot \frac{1}{(2 - 1/x)^2} = \frac{1}{(2x - 1)^2}.$$

2. Rules for exponentials and logarithms. Let a be a positive real number. The basic rules of exponentials are as follows.

Rule 1. If a^b equals B and a^c equals C, then a^{b+c} equals $B \cdot C$, i.e.,

$$a^{b+c} = a^b \cdot a^c.$$

Rule 2. If a^b equals B and B^d equals D, then a^{bd} equals D, i.e.,

$$(a^b)^d = a^{bd}.$$

If a^b equals B, the *logarithm with base a* of B is defined to be b. This is written $\log_a(B) = b$. The function $B \to \log_a(B)$ is defined for all positive real numbers B. Using this definition, the rules of exponentiation become rules of logarithms.

Rule 1. If $\log_a(B)$ equals b and $\log_a(C)$ equals c, then $\log_a(B \cdot C)$ equals $b + c$, i.e.,

$$\log_a(B \cdot C) = \log_a(B) + \log_a(C).$$

Rule 2. If $\log_a(B)$ equals b and B^d equals D, then $\log_a(D)$ equals $d \log_a(B)$, i.e.,

$$\log_a(B^d) = d \log_a(B).$$

Rule 3. Since $\log_B(D)$ equals d, an equivalent formulation is $\log_a(D)$ equals $\log_a(B)\log_B(D)$, i.e.,

$$\log_a(D) = \log_a(B)\log_B(D).$$

3. The derivative of a^x. Let a be a positive real number. What is the derivative of a^x? Denote the derivative of a^x at $x = 0$ by $L(a)$. It equals the value of the limit,

$$L(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Then for every x_0, the derivative of a^x at x_0 equals,

$$\lim_{h \to 0} \frac{a^{x_0+h} - a^{x_0}}{h}.$$

By Rule 1, a^{x_0+h} equals $a^{x_0}a^h$. Thus the limit factors as,

$$\lim_{h \to 0} \frac{a^{x_0}a^h - a^{x_0}}{h} = a^{x_0} \lim_{h \to 0} a^h - 1h.$$

Therefore, for every x, the derivative of a^x is,

$$\frac{d(a^x)}{dx} = L(a)a^x.$$

What is $L(a)$? To figure this out, consider how $L(a)$ changes as a changes. First of all,

$$L(a^b) = \lim_{h \to 0} \frac{(a^b)^h - 1}{h}.$$

By Rule 2, $(a^b)^h$ equals a^{bh}. So the limit is,

$$L(a^b) = \lim_{h \to 0} \frac{a^{bh} - 1}{h} = b \lim_{h \to 0} \frac{a^{bh} - 1}{bh}.$$

Now, inside the limit, make the substitution that k equals bh. As h approaches 0, also k approaches 0. So the limit is,

$$L(a^b) = b \lim_{k \to 0} \frac{a^k - 1}{k} = bL(a).$$

This is very similar to Rule 2 for logarithms.

Choose a number a_0 bigger than 1, say $a_0 = 2$. Then for every positive real number a, $a = a_0^b$ where $b = \log_{a_0}(a)$. Thus,

$$L(a) = L(a_0^b) = bL(a_0) = L(a_0)\log_{a_0}(a).$$

So, with a_0 fixed and a allowed to vary, $L(a)$ is just the logarithm function $\log_{a_0}(a)$ scaled by $L(a_0)$. Looking at the graph of $(a_0)^x$, it is geometrically clear that $L(a_0)$ is positive (though we have not *proved* that $L(a_0)$ is even defined). Thus the graph of $L(a)$ looks qualitatively like the graph of $\log_{a_0}(a)$. In particular, for a less than 1, $L(a)$ is negative. The value $L(1)$ equals 0. And $L(a)$ approaches $+\infty$ and a increases. Therefore, there must be a number where L takes the value 1. By long tradition, this number is called e;

$$L(e) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

This is the definition of e. It sheds very little light on the decimal value of e.

Because e is so important, the logarithm with base e is given a special name: the **natural logarithm**. It is denote by,

$$\ln(a) = \log_e(a).$$

So, finally, $L(a)$ equals,

$$L(a) = \log_e(a)L(e) = \ln(a)(1) = \ln(a).$$

This leads to the formula for the derivative of a^x,

$$\frac{d(a^x)}{dx} = \boxed{\ln(a)a^x.}$$

In particular,

$$\frac{d(e^x)}{dx} = e^x.$$

In fact, e^x is characterized by the property above and the property that e^0 equals 1.

4. The derivative of $\log_a(x)$ and the value of e. By the chain rule,

$$\frac{d(a^u)}{dx} = \ln(a)a^u \frac{du}{dx}.$$

For $u = \log_a(x)$, a^u equals x. Thus,

$$\frac{d(a^u)}{dx} = \frac{d(x)}{dx} = 1.$$

Thus,

$$\ln(a)a^u \frac{du}{dx} = 1.$$

Solving gives,

$$\frac{d\log_a(x)}{dx} = \frac{1}{\ln(a)}\frac{1}{a^u} = \boxed{1/(\ln(a)x)}.$$

In particular, for $a = e$, this gives,

$$\frac{d\ln(x)}{dx} = \boxed{1/x}.$$

What is the derivative of $\ln(x)$ at $x = 1$? On the one hand, since the derivative of $\ln(x)$ equals $1/x$, the derivative at $x = 1$ is $1/1 = 1$. On the other hand, the definition of the derivative gives,

$$\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}.$$

Of course, $\ln(1)$ equals 0, so this simplifies to,

$$\lim_{h\to 0}\frac{1}{h}\ln(1+h).$$

Using Rule 2 for logarithms, this gives,

$$\lim_{h\to 0}\ln((1+h)^{1/h}).$$

Since $\ln(y)$ is continuous, the limit equals,

$$\ln[\lim_{h\to 0}(1+h)^{1/h}].$$

So the natural logarithm of the inner limit equals 1. But e is the unique number whose natural logarithm equals 1. This leads to the formula,

$$e = \lim_{h\to 0}(1+h)^{1/h}.$$

Making the substitution $n = 1/h$ leads to the more familiar form,

$$\lim_{n\to+\infty}(1+1/n)^n = \boxed{e}.$$

This can be used to compute e to arbitrary accuracy. The first few digits of e are 2.718281828459045...

5. Logarithmic differentiation. There is a method of computing derivatives of products of functions that is often useful. If y is a product of n factors, say $f_1(x)\cdot f_2(x)\cdot\dots\cdot f_n(x)$, the derivative of y can be computed by the product rule. However, it seems to be a fact that multiplication is more error-prone than addition. Thus introduce,

$$u = \ln(y) = \ln(f_1(x)) + \ln(f_2(x)) + \cdots + \ln(f_n(x)).$$

The derivative of u is,

$$\frac{du}{dx} = \frac{d}{dx}(\ln(f_1(x))) + \cdots + \frac{d}{dx}(\ln(f_n(x))).$$

Using the chain rule, this is,

$$\frac{du}{dx} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

Thus, far fewer multiplications are needed to compute u'. This is good, because also,

$$\frac{du}{dx} = \frac{d\ln(y)}{dx} = \frac{1}{y}\frac{dy}{dx}.$$

Therefore the derivative of y can be computed as,

$$y' = yu' = (f_1(x) \cdot \cdots \cdot f_n(x))\left(\frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right).$$

Example. Let y be,

$$\frac{(1+x^3)(1+\sqrt{x})}{x^{3/7}}.$$

Then,

$$u = \ln(y) = \ln(1+x^3) + \ln(1+\sqrt{x}) - \frac{3}{7}\ln(x).$$

By the chain rule, $\ln(1+x^3)' = 3x^2/(1+x^3)$ and $\ln(1+\sqrt{x})' = (\sqrt{x})'/(1+\sqrt{x}) = (1/2x^{-1/2})/(1+\sqrt{x})$. Thus, u' equals,

$$u' = \frac{3x^2}{(1+x^3)} + \frac{1}{2\sqrt{x}(1+\sqrt{x})} - \frac{3}{7x}.$$

So, finally,

$$y' = yu' = \frac{(1+x^3)(1+\sqrt{x})}{x^{3/7}}\left(\frac{3x^2}{(1+x^3)} + \frac{1}{2\sqrt{x}(1+\sqrt{x})} - \frac{3}{7x}\right).$$