Lecture 3. September 13, 2005

Homework. Problem Set 1 Part I: (i) and (j).

Practice Problems. Course Reader: 1E-1, 1E-3, 1E-5.

1. Another derivative. Use the 3-step method to compute the derivative of $f(x) = 1/\sqrt{3x+1}$ is,

$$f'(x) = \boxed{-3(3x+1)^{-3/2}/2}.$$

Upshot: Computing derivatives by the definition is too much work to be practical. We need general methods to simplify computations.

2. The binomial theorem. For a positive integer n, the *factorial*,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1,$$

is the number of ways of arranging n distinct objects in a line. For two positive integers n and k, the *binomial coefficient*,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots 3 \cdot 2 \cdot 1},$$

is the number of ways to choose a subset of k elements from a collection of n elements. A fundamental fact about binomial coefficients is the following,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This is known as Pascal's formula. This link is to a webpage produced by MathWorld, part of Wolfram Research.

The *Binomial Theorem* says that for every positive integer n and every pair of numbers a and b, $(a+b)^n$ equals,

$$a^n + na^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + nab^{n-1} + b^n.$$

This is proved by *mathematical induction*. First, the result is very easy when $n = 1$; it just says that $(a+b)^1$ equals $a^1 + b^1$. Next, make the *induction hypothesis* that the theorem is true for the integer n. The goal is to deduce the theorem for $n+1$,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^n b + \cdots + \binom{n+1}{k}a^{n+1-k}b^k + \cdots + (n+1)ab^n + b^{n+1}.$$

By the definition of the $(n+1)^{\text{st}}$ power of a number,

$$(a+b)^{n+1} = (a+b) \times (a+b)^n.$$

By the induction hypothesis, the second factor can be replaced,

$$(a+b)(a+b)^n = (a+b)\left(a^n + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + b^n\right).$$

Multiplying each term in the second factor first by a and then by b gives,

$$\begin{array}{ccccccccccccccc} a^{n+1} & + & na^n b & + & \ldots & + & \binom{n}{k}a^{n+1-k}b^k & + & \binom{n}{k+1}a^{n-k}b^{k+1} & + & \ldots & + & ab^n & & \\ & + & a^n b & + & \ldots & + & \binom{n}{k-1}a^{n+1-k}b^k & + & \binom{n}{k}a^{n-k}b^{k+1} & + & \ldots & + & nab^n & + & b^{n+1} \end{array}$$

Summing in columns gives,

$$a^{n+1} \quad + \quad (n+1)a^n b \quad + \quad \ldots \quad + \quad (\tbinom{n}{k} + \tbinom{n}{k-1})a^{n+1-k}b^k \quad + \quad (\tbinom{n}{k+1} + \tbinom{n}{k})a^{n-k}b^{k+1} \quad + \quad \ldots \quad + \quad (1+n)ab^n$$

Using Pascal's formula, this simplifies to,

$$a^{n+1} + (n+1)a^n b + \dots + \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{k+1} a^{n-k} b^{k+1} + \dots + (n+1)ab^n + b^{n+1}.$$

This proves the theorem for $n+1$, assuming the theorem for n.

Since we proved the theorem for $n = 1$, and since we also proved that for each integer n, the theorem for n implies the theorem for $n+1$, the theorem holds for every integer.

3. The derivative of x^n. Let $f(x) = x^n$ where n is a positive integer. For every a and every h, the binomial theorem gives,

$$f(a+h) = (a+h)^n = a^n + na^{n-1}h + \dots + \binom{n}{k} a^{n-k} h^k + \dots + h^n.$$

Thus, $f(a+h) - f(a)$ equals,

$$(a+h)^n - a^n = na^{n-1}h + \dots + \binom{n}{k} a^{n-k} h^k + \dots + h^n.$$

Thus the difference quotient is,

$$\frac{f(a+h) - f(a)}{h} = na^{n-1} + \binom{n}{2} a^{n-2} h + \dots + \binom{n}{k} a^{n-k} h^{k-1} + \dots + h^{n-1}.$$

Every summand except the first is divisible by h. The limit of such a term as $h \to 0$ is 0. Thus,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = na^{n-1} + 0 + \dots + 0 = na^{n-1}.$$

So $f'(x)$ equals nx^{n-1}.

3. Linearity. For differentiable functions $f(x)$ and $g(x)$ and for constants b and c, $bf(x) + cg(x)$ is differentiable and,

$$(bf(x) + cg(x))' = bf'(x) + cg'(x).$$

This is often called *linearity* of the derivative.

4. The Leibniz rule/Product rule. For differentiable functions $f(x)$ and $g(x)$, the product $f(x)g(x)$ is differentiable and,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

The crucial observation in proving this is rewriting the increment of $f(x)g(x)$ from a to $a+h$ as,

$$f(a+h)g(a+h) - f(a)g(a) = f(a+h)[g(a+h) - g(a)] + f(a+h)g(a) - f(a)g(a) = f(a+h)[g(a+h) - g(a)] + [f(a+h) -$$

5. The quotient rule. Let $f(x)$ and $g(x)$ be differentiable functions. If $g(a)$ is nonzero, the quotient function $f(x)/g(x)$ is defined and differentiable at a, and,

$$(f(x)/g(x))' = [f'(x)g(x) - f(x)g'(x)]/g(x)^2.$$

One way to deduce this formula is to set $q(x) = f(x)/g(x)$ so that $f(x) = q(x)g(x)$, and the apply the Leibniz formula to get,

$$f'(x) = q'(x)g(x) + q(x)g'(x) = q'(x)g(x) + f(x)g'(x)/g(x).$$

Solving for $q'(x)$ gives,

$$q'(x) = [f'(x) - f(x)g'(x)/g(x)]/g(x) = [f'(x)g(x) - f(x)g'(x)]/g(x)^2.$$

6. Another proof that $d(x^n)/dx$ equals nx^{n-1}. This was mentioned only very briefly. The product rule also gives another induction proof that for every positive integer n, $d(x^n)/dx$ equals nx^{n-1}. For $n = 1$, we proved this by hand. Let n be some specific positive integer, and make the induction hypothesis that $d(x^n)/dx$ equals nx^{n-1}. The goal is to deduce the formula for $n + 1$,

$$\frac{d(x^{n+1})}{dx} = (n+1)x^n.$$

By the Leibniz rule,

$$\frac{d(x^{n+1})}{dx} = \frac{d(x \times x^n)}{dx} = \frac{d(x)}{dx}x^n + x\frac{d(x^n)}{dx} = (1)x^n + x\frac{d(x^n)}{dx}.$$

By the induction hypothesis, the second term can be replaced,

$$\frac{d(x^{n+1})}{dx} = x^n + x(nx^{n-1}) = x^n + nx^n = (n+1)x^n.$$

Thus the formula for n implies the formula for $n + 1$. Therefore, by mathematical induction, the formula holds for every positive integer n.