Lecture 29. December 2, 2005

Homework. Problem Set 8 Part I: (c), (d) and (e); Part II: Problems 1 and 2.

Practice Problems. Course Reader: 6B-7.

1. A problem with Riemann integrals. Riemann integrals are defined in very many cases. The result we use most often is that for a piecewise continuous function $f(x)$ on a bounded interval $[a, b]$, the Riemann integral,

$$\int_a^b f(x)dx,$$

exists (and equals a finite number). What if the interval is unbounded, e.g., $[a, \infty)$? Quite simply, the Riemann integral is not defined. This isn't a problem with our methods for computing integrals. It is a problem with the very definition of the Riemann integral. In fact, this is only the first of many problems with the definition of the Riemann integral. Eventually these problems led mathematicians to develop a better definition, the *Lebesgue integral*, which is studied in course 18.103. Luckily, the particular problem of defining the integral on unbounded intervals can be easily overcome using limits (with no need to use the Lebesgue integral).

2. Improper integrals of the first kind. Let $f(x)$ be defined on the interval $[a, \infty)$. If for every number $t > a$ the function $f(x)$ is Riemann integrable on $[a, t]$, and if the limit,

$$\lim_{t \to \infty} \int_a^t f(x)dx,$$

exists, then we say the *improper integral*,

$$\int_a^\infty f(x)dx,$$

is defined and its value is,

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx.$$

Please note, this is a *new* definition. It is not a theorem about Riemann integrals.

Example. Let $p > 1$ be a real number. Then for every $t > 1$, the integral,

$$\int_1^t \frac{1}{x^p}dx,$$

exists and equals,

$$\left(-\frac{1}{(p-1)x^{p-1}}\right|_1^t = \frac{1}{p-1} - \frac{1}{(p-1)t^{p-1}}.$$

Since p is greater than 1, the limit,

$$\lim_{t \to \infty} \frac{1}{t^{p-1}},$$

exists and equals 0. Therefore,

$$\lim_{t \to \infty} \int_1^t \frac{1}{x^p}dx,$$

exists and equals,

$$\frac{1}{p-1}.$$

Therefore the improper integral exists and equals,

$$\int_1^\infty \frac{1}{x^p}dx = \boxed{1/(p-1).}$$

On the other hand, when p equals 1, then,

$$\int_1^t \frac{1}{x}dx = \ln(t).$$

Since the limit $\lim_{t\to\infty} \ln(t)$ is not defined (or more precisely, equals $+\infty$), the improper integral,

$$\int_1^\infty \frac{1}{x}dx,$$

is not defined (or more precisely, equals $+\infty$).

Example. For $t > 0$, the integral,

$$\int_0^t \cos(x)dx,$$

exists and equals $\sin(t)$. Even though all values $\sin(t)$ are defined and bounded, the limit,

$$\lim_{t\to\infty} \sin(t),$$

is not defined (essentially because it never settles down). Therefore the improper integral,

$$\int_0^\infty \cos(x)dx,$$

is not defined.

3. Improper integrals of the second kind. Here is a second problem with the Riemann integral. Let $[a, b]$ be a bounded interval. Let $f(x)$ be a function that is bounded on $[t, b]$ for every $a < t < b$, but which is unbounded on $[a, b]$. According to the definition of the Riemann integral,

$$\int_a^b f(x)dx,$$

is not defined. However, it may happen that for every $a < t < b$, the integral,

$$\int_t^b f(x)dx,$$

is defined and the limit,

$$\lim_{t\to a^+} \int_t^b f(x)dx,$$

is defined. In this case, we say the *improper integral*,

$$\int_{a^+}^b f(x)dx,$$

is defined and its value is,

$$\int_{a^+}^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx.$$

Similarly, if $f(x)$ is Riemann integrable on every interval $[a,t]$ for $a < t < b$, and if

$$\lim_{t \to b^-} \int_a^t f(x)dx,$$

exists, we say the *improper integral*,

$$\int_a^{b-} f(x)dx,$$

exists and its value is,

$$\int_a^{b-} f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx.$$

Example. Let p be a real number in the range $0 < p < 1$. Because the function $1/x^p$ is unbounded on $[0,1]$, the Riemann integral,

$$\int_0^1 \frac{1}{x^p}dx,$$

is not defined. However, for every $0 < t < 1$, the Riemann integral,

$$\int_t^1 \frac{1}{x^p}dx,$$

is defined equals,

$$\frac{1 - t^{1-p}}{1-p}.$$

Since $0 < p < 1$, the limit,

$$\lim_{t \to 0} t^{1-p},$$

exists and equals 0. Therefore,

$$\lim_{t \to 0} \int_t^1 \frac{1}{x^p}dx,$$

exists and equals $1/(1-p)$. Therefore the improper integral,

$$\int_{0+}^1 \frac{1}{x^p}dx,$$

exists and its value is,

$$\int_{0+}^1 \frac{1}{x^p}dx = \boxed{1/(1-p).}$$

4. The Comparison Test. When is an improper integral defined? This is equivalent to asking when a limit is defined. Therefore, every rule for convergence of a limit gives a rule for convergence of an improper integral. There are 2 basic rules for convergence of a limit.

The squeezing lemma. If $F(x) \leq G(x) \leq H(x)$ on an interval, if $\lim_{x \to a} F(x)$ and $\lim_{x \to a} H(x)$ exist, and if $\lim_{x \to a} F(x)$ equals $\lim_{x \to a} H(x)$, then $\lim_{x \to a} G(x)$ exists and equals the other 2 limits.

Monotone bounded limits. If $F(x)$ is monotone increasing and bounded above on $[a, b)$, then $\lim_{x \to b^-} F(x)$ exists. Similarly, if $F(x)$ is monotone decreasing and bounded below, then $\lim_{x \to b^-} F(x)$ exists, if $F(x)$ is monotone increasing and bounded below, then $\lim_{x \to a^+} F(x)$ exists, and if $F(x)$ is monotone decreasing and bounded above, then $\lim_{x \to a^+} F(x)$ exists.

These give the following tests for convergence of an improper integral.

Squeezing lemma. If $f(x) \leq g(x) \leq h(x)$ on the interval $[a, \infty)$, and if the improper integrals,

$$\int_a^\infty f(x)dx \text{ and } \int_a^\infty h(x)dx,$$

exist and are equal, then the improper integral,

$$\int_a^\infty g(x)dx,$$

exists and equals the other 2.

The comparison theorem. If $0 \leq f(x) \leq g(x)$ on $[a, \infty)$, and if,

$$\int_a^\infty g(x)dx,$$

converges, then

$$\int_a^\infty f(x)dx,$$

converges. Contrapositively, if $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.