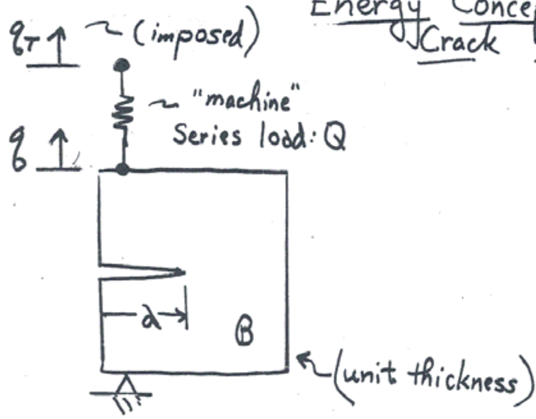
DEPT. OF MATERIALS SCIENCE

3.35 FRACTURE AND FATIGUE

Guest lecture by Prof. David M. Parks

October 2, 2003

Energy Concepts in Linear Elastic Crack Analysis



- **Kinematics:**

$$q_T = q + q_m$$

$$\boxed{dq_T = dq + dq_m}$$

- **Energy**
  - Potential energy: $\pi = \hat{\pi}(q_T, a)$
  - "Machine" Strain energy: U
  - "Body" Strain energy: W

$$\pi = U + W$$

$$\boxed{d\pi = dU + dW}$$

- **Energy Equivalence** (e.g., Virtual Work)

$$U = \int_0^{q_m} Q(q_m')\, dq_m' \;;\Rightarrow \boxed{dU = Q\, dq_m}$$

$$W = \int_0^{q} Q(q', a)\, dq' \;;\Rightarrow \boxed{dW = Q\, dq + \left[\int_0^{q} \frac{\partial Q(q', a)}{\partial a}\, dq'\right] da}$$

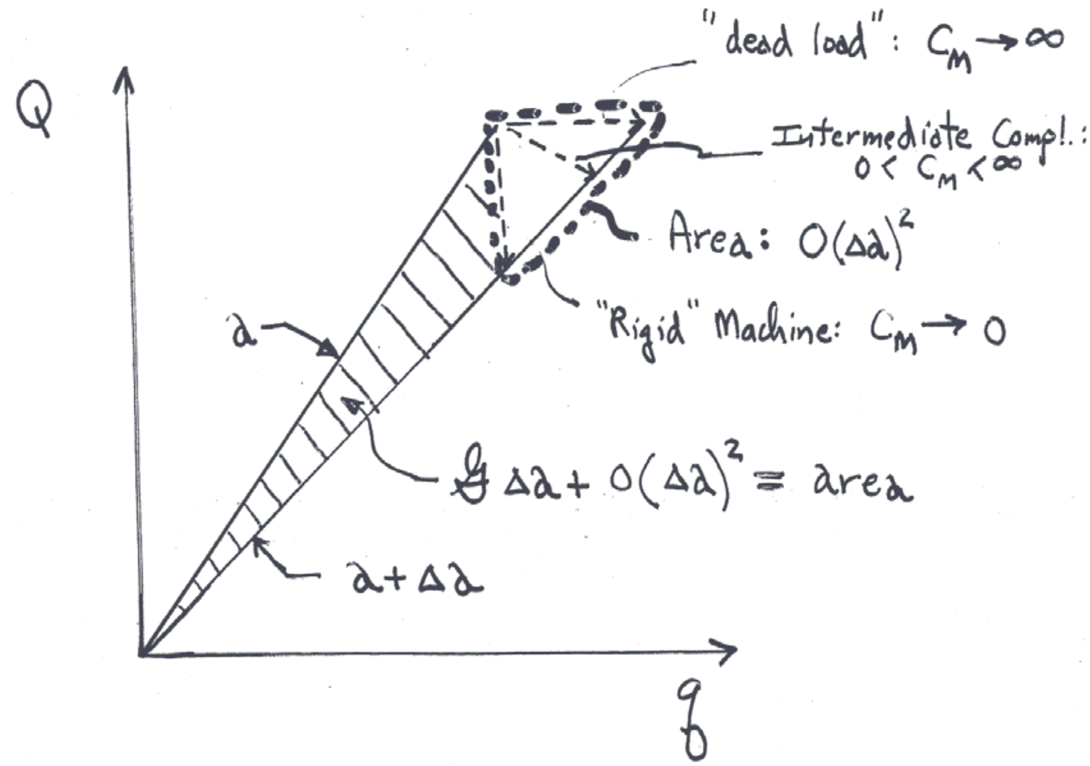
- **Combine Terms**

$$d\pi = Q\underbrace{(dq + dq_m)}_{dq_T} + \left[\int_0^{q} \frac{\partial Q}{\partial a}\, dq'\right] da$$

$$\Rightarrow$$

$$\boxed{\mathcal{G} \equiv -\left.\frac{\partial \pi}{\partial a}\right|_{q_T} = -\int_0^{q} \frac{\partial Q(q', a)}{\partial a}\, dq'}$$

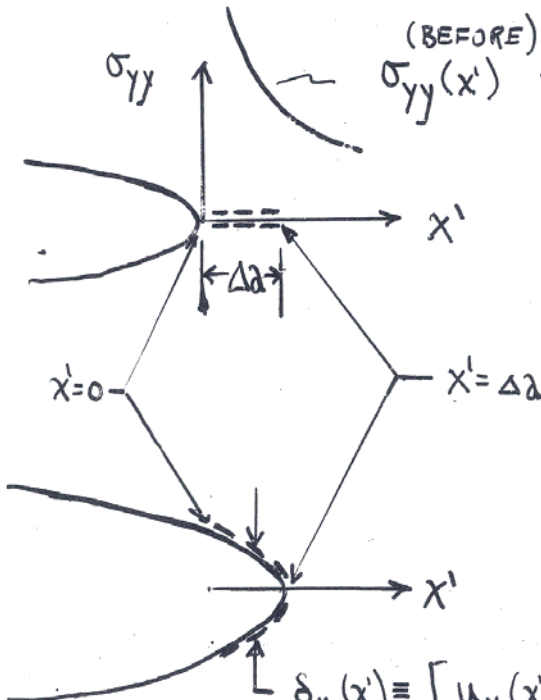
Machine Compliance: $dq_M = C_M \, dQ$



Strain Energy Difference: A Special Path

(BEFORE)

$$\sigma_{yy}^{(BEFORE)}(x') = \frac{K_I}{\sqrt{2\pi x'}}$$



(AFTER)

$$\delta_y(x') \equiv \left[u_y(x', y=0^+) - u_y(x', y=0^-)\right]^{(AFTER)}$$

$$= \frac{8\,(K_I + \Delta K_I)}{E'} \cdot \left(\frac{\Delta a - x'}{2\pi}\right)^{1/2}$$

WORK DONE During Proportional Traction Drop

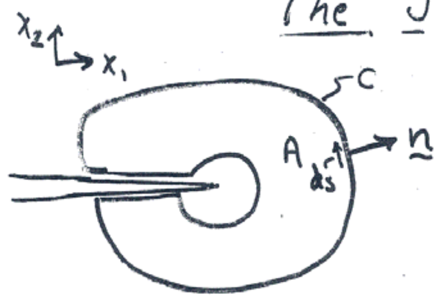
$$\Delta W = -\frac{1}{2}\int_0^{\Delta a} \sigma_{yy}^{(BEFORE)}(x')\; \delta_y(x')\, dx'$$

$$\boxed{\mathcal{G} = \lim_{\Delta a \to 0} \frac{\Delta W}{\Delta a} = \frac{K_I^2}{E'}}$$

MIXED MODE:

$$\mathcal{G} = \frac{K_I^2 + K_{II}^2}{E'} + K_{III}^2/2G$$

The J-Integral



Assumptions (Relaxable / Generalizable)

- Linear elasticity / quasistatic
- No body forces
- Traction-free Crack Faces

$$W(\underset{\sim}{\varepsilon}) = \int_0^{\underset{\sim}{\varepsilon}} \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}')\cdot d\underset{\sim}{\varepsilon}' = \frac{1}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}$$

Let $$I \equiv \oint_C \left\{ W n_1 - \sigma_{ij} n_j u_{i,1} \right\} ds$$

$$= \oint_C \left\{ W \delta_{1j} - \sigma_{ij} u_{i,1} \right\} n_j \, ds$$

(Divergence Thm)

$$= \iint_A \frac{\partial}{\partial x_j}\left\{ W \delta_{1j} - \sigma_{ij} u_{i,1} \right\} dA$$

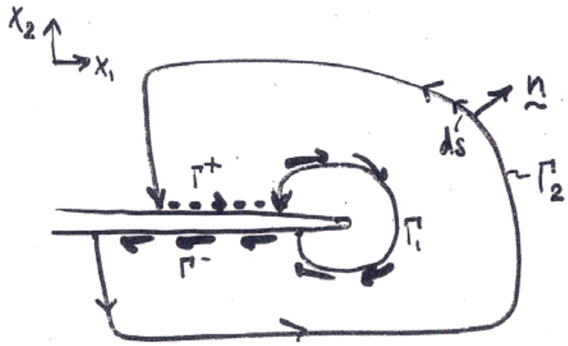
- $$\delta_{1j}\frac{\partial W}{\partial x_j} = \frac{\partial W}{\partial \varepsilon_{mn}}\delta_{1j}\frac{\partial \varepsilon_{mn}}{\partial x_j} = \delta_{1j}\sigma_{mn}\frac{\partial \varepsilon_{mn}}{\partial x_j} = \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}$$

ADD TO ZERO

- $$-\frac{\partial}{\partial x_j}\left(\sigma_{ij} u_{i,1}\right) = -\left\{ \overset{0}{\sigma_{ij,j}}\, u_{i,1} + \sigma_{ij}\, u_{i,1j} \right\} = -\underbrace{\sigma_{ij}\, u_{i,j1}}_{\text{sym}}$$
$$= -\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}$$

$$\therefore \quad \boxed{I \equiv 0}$$

Path-Independence of J



Let $$J(\Gamma) \equiv \int_\Gamma \left\{ \underline{W n_1 - \sigma_{ij} n_j u_{i,1}} \right\} ds$$

for any $\overset{ccw}{\wedge}$ path Γ starting on lower face and terminating on top face...

Path-Independence: $J(\Gamma_1) = J(\Gamma_2) = J$

Proof: $$I \equiv \oint_C \{-\} ds = 0 = \int_{\Gamma^- + \Gamma_2 + \Gamma^+ + (-\Gamma_1)} \{-\} ds$$

But: $\int_{\Gamma^+} \{-\} ds = \int_{\Gamma^-} \{-\} ds = 0$ because $n_1 = 0$ & $t_i = \sigma_{ij} n_j = 0$

AND $\int_{(-\Gamma_1)} \{-\} ds = -\int_{\Gamma_1} \{-\} ds \Rightarrow \boxed{\int_{\Gamma_1} \{-\} ds = \int_{\Gamma_2} \{-\} ds \equiv J}$

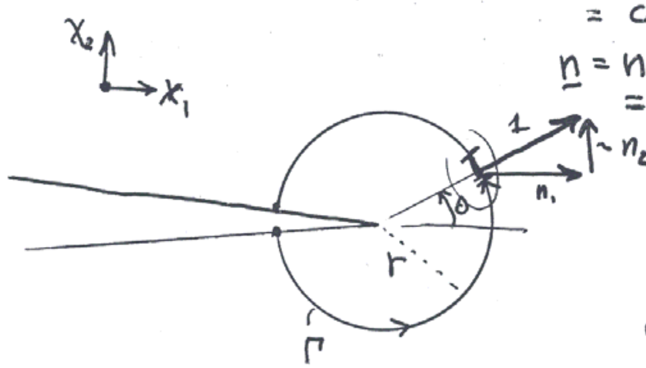
A Special Contour for J Integral:

a Circle of Radius "r" Centered at Crack Tip....

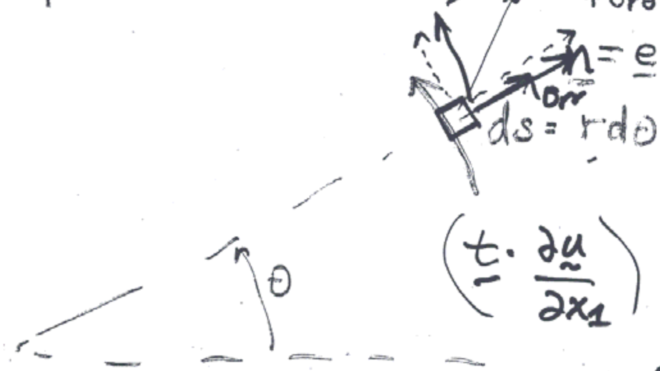
$$\underset{\sim}{n} = n_1 \underset{\sim}{e}_1 + n_2 \underset{\sim}{e}_2 = \cos\theta\, \underset{\sim}{e}_1 + \sin\theta\, \underset{\sim}{e}_2 = \underset{\sim}{e}_r(\theta)$$

$$= \underset{\sim}{e}_r$$

$$n_1 = \cos\theta$$



$$\underset{\sim}{t} = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \sigma_{rr} \underset{\sim}{e}_r + \sigma_{r\theta} \underset{\sim}{e}_\theta$$



$$\left(\underset{\sim}{t} \cdot \frac{\partial \underset{\sim}{u}}{\partial x_1}\right)$$

$$\underset{\sim}{u} = u_r \underset{\sim}{e}_r + u_\theta \underset{\sim}{e}_\theta$$

$$\therefore J = \int_\Gamma \left\{ W n_1 - \underbrace{(n_j \sigma_{ji})}_{t_i} \underbrace{u_{i,1}}_{u_{i,1}} \right\} ds$$

$$J = r \int_{-\pi}^{\pi} \left\{ W(r,\theta) \cos\theta - \left(\sigma_{rr} \frac{\partial u_r}{\partial x_1} + \sigma_{r\theta} \frac{\partial u_\theta}{\partial x_1} \right) \right\} d\theta$$

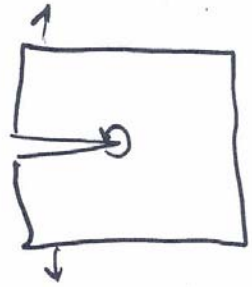
result: J indep. of 'r' $\Rightarrow$ integrand $\sim \frac{1}{r}$ as $r \to 0$

$$\left.\begin{aligned} W &\sim \int \sigma \cdot d\epsilon \sim \sigma \cdot \epsilon \\ \sigma \cdot \frac{\partial u}{\partial x} &\sim \sigma \cdot \epsilon \end{aligned}\right\} \Rightarrow$$ Product of $\sigma \cdot \varepsilon \sim 1/r$ as $r \to 0$!

Linear Elasticity:

$$\boxed{J = \mathcal{G} = K_I^2 / E'}$$

- Method 1: insert asymptotic K_I-fields of σ_{ij}, ε_{ij}, u_i, and $\mathcal{W} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ into J integrand; evaluate on circle of radius "r"



$$\mathcal{W} \sim \sigma \cdot \varepsilon \sim \left(\frac{K}{\sqrt{r}}\right)\left(\frac{K}{E\sqrt{r}}\right) \sim \frac{K^2}{Er}$$

"$\frac{1}{r}$" cancels with $ds = r\,d\theta$

$$J = \int \underbrace{ds}_{r\,d\theta}\,(\text{——}) \Rightarrow J = K_I^2 / E'$$

- Method 2

Line integral J <u>directly</u> evaluates $\mathcal{G}$ (energy flux) (Rice);

$$J = \mathcal{G} = (\text{previous}: K_I^2 / E')$$

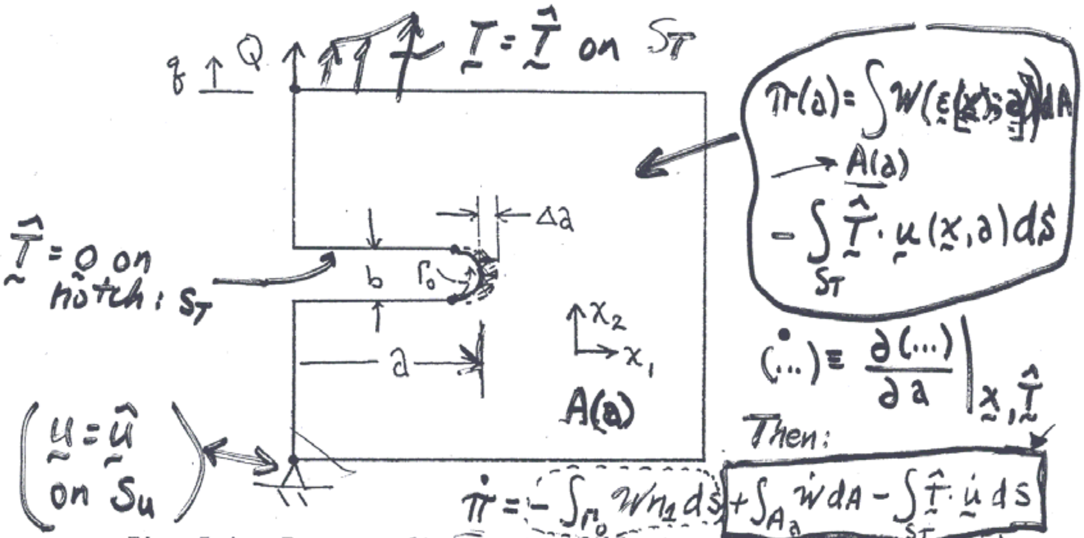


Fig. 7.4: Energy release rate of blunt notch of depth a, a + Δ a.

SUM TO ZERO

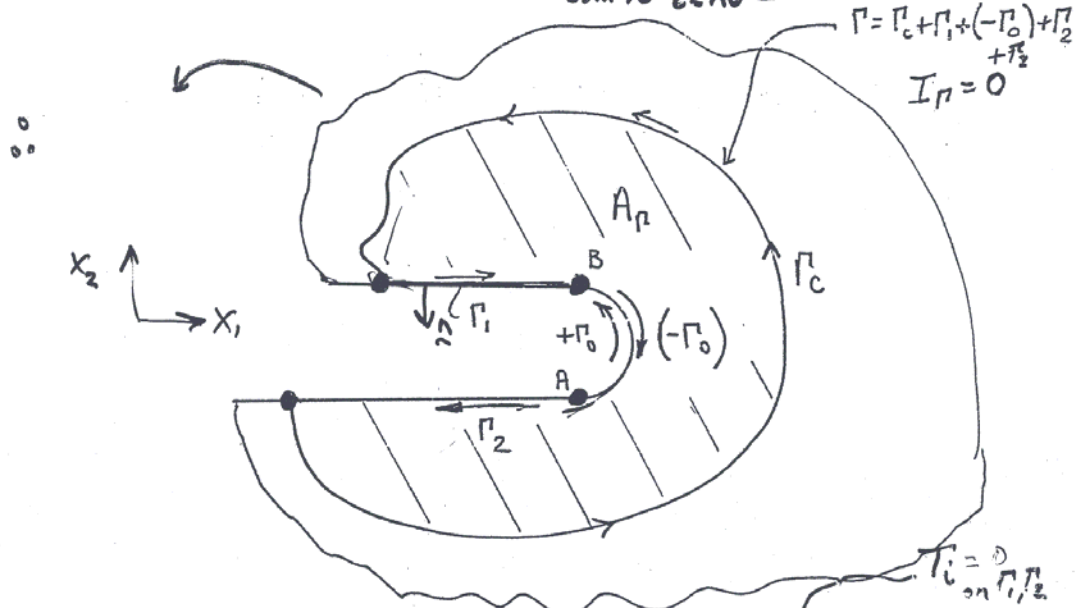


Fig. 7.5: A closed contour $\Gamma = \Gamma_c + \Gamma_1 + (-\Gamma_0) + \Gamma_2$ over which I_Γ vanishes

$$0 = I_\Gamma = \int_\Gamma \left\{ W n_1 - (\sigma_{ij} n_j) \frac{\partial u_i}{\partial x_1} \right\} ds$$
$$= \cancel{\int_{\Gamma_1}} + \cancel{\int_{\Gamma_2}} + \left(-\int_{\Gamma_0}\right) + \int_{\Gamma_c} = 0$$

$$\int_{A(a)} \dot{W}\,dA \overset{(\text{Definition } \dot{W})}{=} \int_{A(a)} \sigma_{ij}\,\dot{\varepsilon}_{ij}\,dA$$

PRINCIPAL OF Virtual Work (Power)

$$= \int_{S = S_T + S_u} n_j\,\sigma_{ji}\,\dot{u}_i\,dS$$

$(\hat{T}_j = n_i\,\sigma_{ij} \text{ on } S_T)$

$$= \int_{S_T} \hat{\underset{\sim}{T}} \cdot \dot{\underset{\sim}{u}}\,ds + \int_{S_u} \overset{0}{\dot{\underset{\sim}{u}}_i}\, n_j\,\sigma_{ji}\,dS$$

$\underset{\sim}{u} = \hat{\underset{\sim}{u}}$ on S_u;

$\dot{\underset{\sim}{u}} = \underset{\sim}{0}$

$$\int_{A(a)} \dot{W}\,dA = \int_{S_T} \hat{\underset{\sim}{T}} \cdot \dot{\underset{\sim}{u}}\,ds$$

$\therefore$ Last 2 terms cancel, & ...

$$-\dot{\pi} = \int_{\Gamma_0} \Big[W n_1 - \overbrace{n_j\,\sigma_{ji}}^{\text{Traction free on } \Gamma_0 \Rightarrow \,=\, \underset{\sim}{0}}\, u_{i,1} \Big]\,ds$$

add zero term to $-\dot{\pi}$

$$I_n = 0 = -\int_{\Gamma_0} \left(W n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) ds + \int_{\Gamma_c} (\ldots) \, ds$$

Finally, note

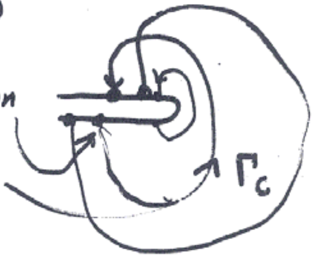
$t_j = 0$

$$-\dot{\pi} = \int_{\Gamma_0} \left\{ W n_1 - \underbrace{(n_i \sigma_{ij})}_{= 0 \text{ on } \Gamma_0} u_{j,1} \right\} ds = \int_{\Gamma_c} ds \left\{ W n_1 - n_i \sigma_{ij} u_{j,1} \right\}$$

(because closed contour = 0)

$$-\frac{\partial \pi}{\partial a} = -\dot{\pi} = \int_{\Gamma_c} \left\{ W n_1 - n_i \sigma_{ij} u_{j,1} \right\} ds$$

Γ_c — any contour starting on and ending here

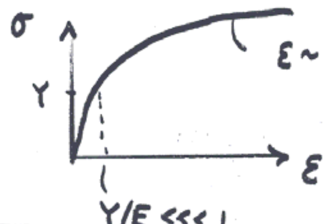


$$\therefore \; -\dot{\pi} \equiv J = \int_{\Gamma_c} \left\{ W n_1 - n_i \sigma_{ij} u_{j,1} \right\} ds$$

Obvious extensions: surface tractions, body forces ⇒ 'area' terms or crack face traction contributions to integral...

The utility of (the numerical value of) a conservation integral such as J in the interpretation of fracture must rest on an ~unique relation between the value and aspects of the crack tip field relevant to crack extension....

Power Law Materials / HRR Fields



$\varepsilon \sim \sigma^n$ for "large" (w.r.t Y/E) ε
(in tension)

CRACK TIP

Choose Γ_c as circle of radius r

each term $O \sim \sigma \cdot \varepsilon$; must have $\sigma \cdot \varepsilon \sim \frac{1}{r}$

Generally finite $\neq 0$ "SMALL"

$$J = r \int_{-\pi}^{\pi} d\theta \left\{ W\cos\theta - \sigma_{rr}\frac{\partial u_r}{\partial x_1} - \sigma_{r\theta}\frac{\partial u_\theta}{\partial x_1} \right\}$$

quickly: suppose $\sigma \sim r^p$

$$\varepsilon \sim \sigma^n \sim (r^p)^n = r^{np}$$

$$\sigma \cdot \varepsilon \sim r^{np+p} = r^{-1}$$

$$\Rightarrow \boxed{p = -1/(n+1)}$$