3.185 Test 1

Diffusion, Heat Conduction

Solutions

1. Write your name on all of your answer booklets

   Would you believe someone got this wrong last year? Like I've been saying, this year's class is so much better...

2. Macromolecule diffusion into muscle tissue

   (a) Start with the unsteady spherical diffusion equation from the equation sheet:

   $$\frac{\partial C}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D \frac{\partial C}{\partial r}\right) + G.$$

   Steady-state implies $\partial C/\partial t$ is zero, and without other information, we can ignore generation, leaving:

   $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial C}{\partial r}\right) = 0.$$

   Cancel the $1/r^2$ up front, and substitute $C = A/r + B$, and simplify:

   $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{A}{r} + B\right)\right) = \frac{\partial}{\partial r}\left(r^2\left(-\frac{A}{r^2} + 0\right)\right) = \frac{\partial}{\partial r}(-A) = 0.$$

   So it works.

   (b) Substitute the boundary conditions into $C = A/r + B$; at $r = \infty$, $C = 0$:

   $$0 = \frac{A}{\infty} + B \Rightarrow B = 0,$$

   then at $r = 0.5\text{mm}$, $C = C_0$:

   $$C_0 = \frac{A}{0.5\text{mm}} \Rightarrow A = C_0 \cdot 0.5\text{mm}.$$

   The solution which fits these boundary conditions is thus:

   $$C = \frac{C_0 \cdot 0.5\text{mm}}{r}.$$

   (c) Set $C \geq C_0/2$, so $C_0/2 \leq C$, and solve for r:

   $$C_0/2 \leq \frac{C_0 \cdot 0.5\text{mm}}{r}$$

   $$r \leq 2 \cdot 0.5\text{mm} = 1\text{mm}$$

   So $C \geq C_0/2$ where $r \leq 1\text{mm}$, and this region is a sphere with twice the diameter of the device.

(d) In steady state, there will be no change to the size or shape of this region, it will still be a sphere with twice the diameter of the device.

This is a bit counter-intuitive, as the initial unsteady behavior will be quite different, the drug will spread rapidly along the direction of the cells and slowly across them. But steady-state will be rapidly reached in that longitudinal direction, then slowly in the transverse direction, eventually becoming spherically symmetrical like the isotropic case.

One way to think of this is that the r-direction diffusivity D_{rr} depends on the angle ϕ away from the direction of muscle cells (where the angles $\phi = 0$ and $\phi = \pi$ (180°) are parallel to the cells, and $\phi = \pi/2$ (90°) is perpendicular to the cells, but not on θ (longitude-type angle around the cells) or r. So parallel to the cells, $D_{rr}(\phi = 0) = D_{rr}(\phi = \pi)$ is large; perpendicular to them, $D_{rr}(\phi = \pi/2)$ is small; at a 45°angle, $D_{rr}(\phi = \pi/4)$ is somewhere between the other two.

Then if we try $C = A/r + B$ in the modified diffusion equation, it still works:

$$\frac{\partial}{\partial r}\left(r^2 D_{rr}(\phi)\frac{\partial C}{\partial r}\right) = \frac{\partial}{\partial r}\left(r^2 D_{rr}(\phi)\left(-\frac{A}{r^2}+0\right)\right) = \frac{\partial}{\partial r}(D_{rr}(\phi)A) = 0.$$

For those interested in even more details, they go something like this. In an isotropic medium, the diffusivity is a scalar. In an anisotropic medium, it is a tensor, so Fick's first law goes like:

$$J_i = -D_{ij}C_{,j},$$

where D_{ij} is the diffusivity tensor and $C_{,j}$ the concentration gradient (in indicial notation). If we point the z-axis in the direction of the muscle cells, and the x and y axes in orthogonal directions, then the diffusivity tensor will look like:

$$D_{ij} = \begin{pmatrix} D_\perp & 0 & 0 \\ 0 & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix},$$

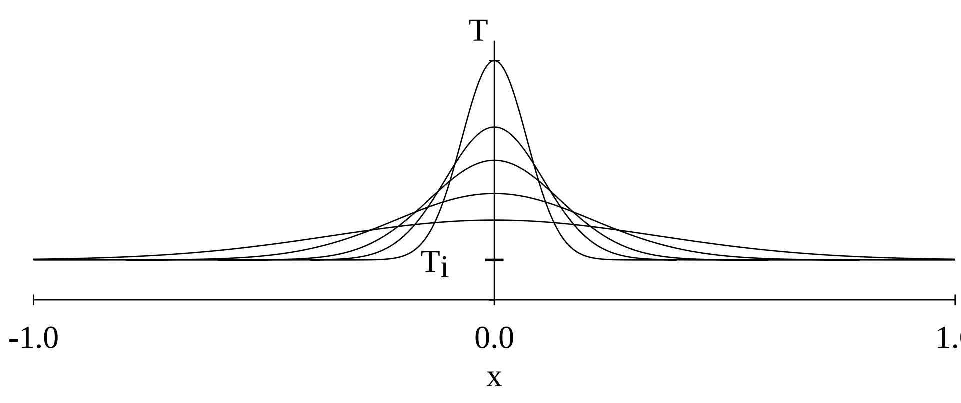
where $D_\perp$ is the low diffusivity across the cells and $D_\parallel$ is the high diffusivity along the cells. So J_z is faster for a given $\partial C/\partial z$ than J_x would be for the same $\partial C/\partial x$.

(e) There are multiple potential complicating factors, any one of which would receive full credit:

- Blood flow carries the drug around, changing the shape significantly, by *convection*.
- The muscle contracts and extends, changing its shape, also effectively a *convection* mechanism.
- The drug is metabolized or otherwise broken down by the body by chemical reactions, macrophages, etc., resulting in nonzero *generation*.
- Non-uniformities in the muscle such as fluid regions, or else muscle damage or scar tissue due to the implantation process, results in *non-uniform diffusivity* (not only anisotropic).

3. Heat Transfer in Resistance Welding

(a) With a fixed amount of excess heat deposited in a thin layer at the junction, and that heat diffusing out along the lengths of the rods, this looks like the shrinking Gaussian:



(b) At very short time scales, we have no idea what the temperature distribution looks like. At moderate to long time scales, we can use the Shrinking Gaussian solution, which was given on the equation sheet:

$$T = T_i + \frac{(T_0 - T_i)\delta}{\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

(c) Okay, so we have the energy per unit cross-section area of weld, and we need something like $T_0 - T_i$ and δ, both of which are unknown. To get there, we can use the relationship between temperature change and volumetric enthalpy density change:

$$\Delta H = \rho c_p \Delta T \Rightarrow \Delta T = \frac{\Delta H}{\rho c_p}$$

Here when we divide enthalpy per unit area by ρc_p, we get something interesting:

$$\frac{3 \times 10^6 \frac{\text{J}}{\text{m}^2}}{2700\frac{\text{kg}}{\text{m}^3} \cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}} = 1.21\text{K} \cdot \text{m}$$

These units $\text{K}\cdot\text{m}$ are exactly the units of $(T_0 - T_i)\delta$. In fact, just as in diffusion the area under the shrinking Gaussian represents the total solute content, which is fixed, the area under the thermal shrinking Gaussian represents the total heat content per unit cross-section area, which is fixed.

Because δ represents half of the thickness of the original heated region in this formulation, we need to use half of the energy for $(T_0 - T_i)\delta$, and can then look for T at $x = 0$:

$$T = 40°\text{C} + \frac{0.606\text{K} \cdot \text{m}}{\sqrt{\pi \cdot \frac{238\frac{\text{W}}{\text{m}\cdot\text{K}}}{2700\frac{\text{kg}}{\text{m}^3}\cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}} \cdot 1\text{second}}} \exp(-0) = 40°\text{C} + 35°\text{C} = 75°\text{C}.$$

(d) The maximum temperature is 35°C above the temperature of the rest of the rod. To calculate the width where temperature difference is at least half that, we need only calculate where:

$$\exp\left(-\frac{x^2}{4Dt}\right) \geq \frac{1}{2}$$

$$x^2 = 4\alpha t \ln(2) \Rightarrow x = 2\sqrt{\frac{238\frac{\text{W}}{\text{m}\cdot\text{K}}}{2700\frac{\text{kg}}{\text{m}^3} \cdot 917\frac{\text{J}}{\text{kg}\cdot\text{K}}} \cdot 1\text{second} \cdot \ln(2)} = 0.016\text{m}.$$

This is the distance along the rod from $x = 0$ where this is satisfied, so the full length of this region in both directions from the weld is twice this, or about 3.2 cm.

4. Time scales (25 pts)

(a) The time scale for diffusive or conductive steady-state is reached when gradients in the concentration or temperature field no longer lead to changes in that field with time.

$$\text{Diffusion}: t_{SS} = \frac{L^2}{D};\ \text{Heat Conduction}: t_{SS} = \frac{L^2}{\alpha},\ \text{where } \alpha = \frac{k}{\rho c_p}.$$

(b) The Biot number is given by kL/D or $h_D L/D$ for diffusion, and hL/k for heat conduction. For a given material (D or conductivity k) and geometry (L), lower Biot number means lower k (reaction rate constant), h_D or h (mass/heat transfer coefficient). This added resistance to heat flow requires longer time to reach steady-state with the surroundings, so lower Biot number implies longer timescale.

(c) First, the Biot number for one-sided cooling is:

$$\text{Bi} = \frac{hL}{k} = \frac{100\frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 0.02\text{m}}{30\frac{\text{W}}{\text{m}\cdot\text{K}}} = 0.067$$

So cooling is convection-limited, and we can assume Newtonian cooling.

The Newtonian cooling was given as:

$$\frac{T - T_{fl}}{T_i - T_{fl}} = \exp\left(-\frac{Aht}{V\rho c_p}\right)$$

The initial and fluid temperatures are $T_i = 425°\text{C}$ and $T_{fl} = 25°\text{C}$, and we want t when $T = 65°\text{C}$, or $\frac{T-T_{fl}}{T_i-T_{fl}} = 0.1$. For one-sided cooling, V/A is just the thickness. Solve for t:

$$t = -\frac{V\rho c_p}{Ah} \ln\left(\frac{T - T_{fl}}{T_i - T_{fl}}\right) = -\frac{0.02\text{m} \cdot 7600\frac{\text{kg}}{\text{m}^3} \cdot 700\frac{\text{J}}{\text{kg}\cdot\text{K}}}{100\frac{\text{W}}{\text{m}^2\cdot\text{K}}} \ln(0.1) = 2450\text{seconds}$$

So it takes a good fraction of an hour to cool down this much.

(d) I'm afraid I messed up slightly when writing this problem, and assumed a Biot number of 4, or in W^3C notation, $m = 0.25$. Which is sort of right, but since the W^3C graphs (rightly, when cooling from both sides) use half the thickness as the length scale x_1 on these graphs, $x_1 = 0.01\text{m}$, and $\text{Bi} = 6000 \times 0.01/30 = 2$, and $m = 0.5$.

Trouble is, the $m = 0.5$ curve doesn't hit $\frac{T-T_\infty}{T_0-T_\infty} = 0.1$ on the provided graph. Fortunately, in the log-linear scale of these plots, the curve is roughly straight for this section (both the Newtonian cooling and $n = 1$ term of the Fourier series are absolutely straight since they go as e^{-t}), so we can pretty safely extrapolate to the dimensionless temperature of 0.1, which lands at a Fourier number of about 1.9:

$$\text{Fo} = X = 1.9 = \frac{\alpha t}{x_1^2}$$

$$t = X\frac{\rho c_p x_1^2}{k} = 1.9\frac{7600\frac{\text{kg}}{\text{m}^3} \cdot 700\frac{\text{J}}{\text{kg}\cdot\text{K}} \cdot (0.01\text{m})^2}{30\frac{\text{W}}{\text{m}\cdot\text{K}}} = 33.7\text{seconds}$$

This is dramatically faster!

These two quench rates, in water and air, illustrate that for a given material (k, ρ, c_p) and geometry (L), lower Biot number results in much longer time scale. (Well, L isn't quite comparable, but the $L/2$ used in this part results in only two-fold reduction in timescale for Newtonian cooling, four-fold for conduction-limited cooling; the much higher h results in another 18-fold reduction!)