3.185 Problem Set 1

Math Review

Solutions

1. Calculate the outer product matrix for the vectors $(10, 5, 6)$ and $(3, 4, 5)$.

$$\begin{pmatrix} 10 \\ 5 \\ 6 \end{pmatrix} (3,4,5) = \begin{pmatrix} 10\cdot 3 & 10\cdot 4 & 10\cdot 5 \\ 5\cdot 3 & 5\cdot 4 & 5\cdot 5 \\ 6\cdot 3 & 6\cdot 4 & 6\cdot 5 \end{pmatrix} = \begin{pmatrix} 30 & 40 & 50 \\ 15 & 20 & 25 \\ 18 & 24 & 30 \end{pmatrix}$$

2. For the time-dependent temperature field:

$$T = 400 - 50z \exp\left(-t - x^2 - y^2\right)$$

   (a) The gradient is:

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$
$$= \left(100xz \exp\left(-t - x^2 - y^2\right), 100yz \exp\left(-t - x^2 - y^2\right), -50 \exp\left(-t - x^2 - y^2\right)\right)$$

   (b) The definition of the substantial derivative is:

$$\frac{DT}{Dt} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) T$$

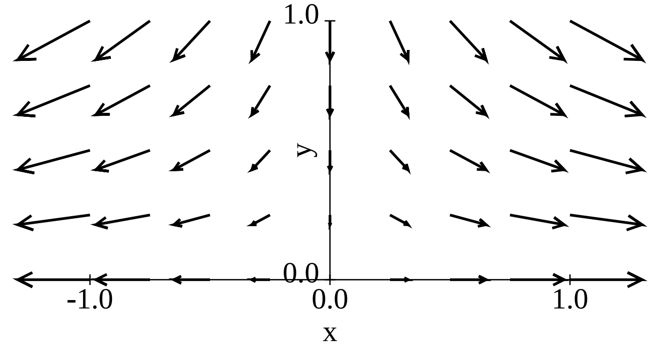
   Since $\vec{u} = 2\hat{\jmath}$ only has a y-component, $\vec{u}\cdot\nabla$ is $u_y \frac{\partial}{\partial y}$, or in this case $2\frac{\partial}{\partial y}$. Therefore:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + 2\frac{\partial T}{\partial y}$$
$$= 50z \exp\left(-t - x^2 - y^2\right) + 200yz \exp\left(-t - x^2 - y^2\right)$$
$$= 50z(1 + 4y) \exp\left(-t - x^2 - y^2\right)$$

3. The velocity field is:

$$u_x = ax,\ u_y = -ay.$$

   (a) For positive y, the vector field looks something like:



(b) The divergence is:

$$\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = a - a = 0.$$

(c) The definition of the curl is:

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

Notice that for 2-D flow, $u_z = 0$ and that the derivatives with respect to z are zero, so only the z-component is non-zero (which is what "Effectively the z-component" referred to). The curl is therefore:

$$\nabla \times \vec{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{k}$$

Both of these partial derivatives are zero, so the curl is zero.

4. For the differential equation:

$$\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} + \frac{dy}{dx} - y = 0$$

(a) We begin by assuming a solution of the form $y = Ae^{Rx}$, taking the derivatives of this gives us the characteristic polynomial:

$$R^3 - R^2 + R - 1 = 0$$

This has three roots at $R = 1$ and $R = \pm i$, giving the solution

$$y = Ae^x + Be^{ix} + Ce^{-ix}$$

Because $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, which are not real but complex, we need to combine solutions to give real ones. If we add them we get $D \cos x$, and subtracting and multiplying by $-i$ gives $E \sin x$ where D and E are new constants. The real solution is therefore written:

$$y = Ae^x + D \cos x + E \sin x$$

Another way to look at this is that we can let A, B and C be complex:

$$A = a_r + a_i i, \;\; B = b_r + b_i i, \;\; C = c_r + c_i i$$

where a_r, a_i, b_r, b_i, c_r and c_i are all real. Then the solution becomes

$$y = (a_r + a_i i)e^x + (b_r + c_r + b_i i + c_i i) \cos x + (b_r i - c_r i - b_i + c_i) \sin x$$

which is real if a_i, $b_i + c_i$ and $b_r - c_r$ are all zero. So D and E above are equal to $b_r + c_r$ and $c_i - b_i$ respectively.

This is one of those things which you do once, and then from then on, just remember that it works. So if you see $\exp(\pm kix)$ as a pair of solutions, for the real part, just substitute $\sin(kx)$ and $\cos(kx)$.

(b) At $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$; at $x = \pi$, $y = -1$. These boundary conditions give us:

$$y = Ae^0 + D \cos 0 + E \sin 0 = A + D = 1, \;\; \frac{dy}{dx} = Ae^0 - D \sin 0 + E \cos 0 = A + E = 1, \;\text{ and}$$

$$y = Ae^\pi + D \cos \pi + E \sin \pi = Ae^\pi - D = -1$$

Adding the first and third gives $A(1 + e^\pi) = 0$ so $A = 0$. With this out of the way, the first and second give $D = 1$ and $E = 1$, so the solution for these boundary conditions is

$$y = \cos x + \sin x$$

5. The error function is defined by:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi,$$

To calculate

$$\frac{\partial}{\partial t}\mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right),$$

start with the chain rule:

$$\frac{d}{dt} f(g(t)) = g'(t) f'(g(t)).$$

Here we can set f to the error function, and $g(t)$ to $\frac{y}{2\sqrt{Dt}}$:

$$\frac{\partial}{\partial t}\mathrm{erf}(g(t)) = g'(t)\frac{2}{\sqrt{\pi}}e^{-g(t)^2}.$$

Then substitute $g(t)$:

$$\frac{\partial}{\partial t}\mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) = \left(-\frac{y}{4\sqrt{\alpha t^3}}\right)\frac{2}{\sqrt{\pi}}e^{-\frac{y^2}{4\alpha t}},$$

and simplify:

$$\frac{\partial}{\partial t}\mathrm{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) = -\frac{y}{2\sqrt{\pi \alpha t^3}}e^{-\frac{y^2}{4\alpha t}}.$$

6. Start with:

$$C = \frac{a}{\sqrt{t}}e^{-\frac{x^2}{4Dt}}.$$

To show that it satisfies

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2},$$

first take derivatives, with respect to time:

$$\frac{\partial C}{\partial t} = -\frac{a}{2t^{3/2}}e^{-\frac{x^2}{4Dt}} + \frac{a}{\sqrt{t}}\frac{x^2}{4Dt^2}e^{-\frac{x^2}{4Dt}}$$

$$\frac{\partial C}{\partial t} = \frac{a}{2t^{5/2}}\left(\frac{x^2}{2D} - t\right)e^{-\frac{x^2}{4Dt}},$$

and with respect to x:

$$\frac{\partial^2 C}{\partial x^2} = \frac{\partial}{\partial x}\left[-\frac{ax}{2Dt^{3/2}}e^{-\frac{x^2}{4Dt}}\right]$$

$$\frac{\partial^2 C}{\partial x^2} = -\frac{a}{2Dt^{3/2}}e^{-\frac{x^2}{4Dt}} + \frac{ax^2}{4D^2t^{5/2}}e^{-\frac{x^2}{4Dt}}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{a}{2Dt^{5/2}}\left(\frac{x^2}{2D} - t\right)e^{-\frac{x^2}{4Dt}}.$$

It's pretty clear that

$$\frac{a}{2t^{5/2}}\left(\frac{x^2}{2D} - t\right)e^{-\frac{x^2}{4Dt}} = D\left[\frac{a}{2Dt^{5/2}}\left(\frac{x^2}{2D} - t\right)e^{-\frac{x^2}{4Dt}}\right],$$

so therefore,

$$C = \frac{a}{\sqrt{t}}e^{-\frac{x^2}{4Dt}} \text{ is a solution to } \frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}.$$

As was said in class, this is just about the hardest math we'll use in 3.185. So if you can handle this much, you're off to a great start!