THE EQUATION OF MOTION IN RECTANGULAR COORDINATES (x, y, z)

In terms of τ:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \qquad (A)$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \qquad (B)$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \qquad (C)$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and μ:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \qquad (D)$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \qquad (E)$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \qquad (F)$$

The three fundamental equations of conservation

	I		II		III		IV			V
EQUATION OF CONSERVATION OF:	Local change	+	Change by convection	+	Change by diffusion	+	Change by production	=	0	Boundary condition
MASS	$\frac{\partial c}{\partial t}$	+	$v\frac{\partial c}{\partial x}$	−	$D\frac{\partial^2 c}{\partial x^2}$	+	r	=	0	Mass transfer $= k_m a \Delta c$
ENERGY	$c_p \rho \frac{\partial T}{\partial t}$	+	$c_p \rho v \frac{\partial T}{\partial x}$	−	$\lambda \frac{\partial^2 T}{\partial x^2}$	+	$\dot{q}$	=	0	Heat transfer $= h a \Delta T$
MOMENTUM	$\rho \frac{\partial v}{\partial t}$	+	$\rho v \frac{\partial v}{\partial x}$	−	$\eta \frac{\partial^2 v}{\partial x^2}$	+	f	=	0	Shear force $= \tau a$ Surface tension force $= \gamma l$

CORRESPONDING QUANTITIES (per unit of volume)	Unit	Diffusive transport	Production	Boundary transfer
MASS	c	D	r	$k_m \Delta c$
ENERGY	$c_p \rho T$	λ	$\dot{q}$	$h \Delta T$
MOMENTUM	ρv	η	f	τ or γL^{-1}

System of dimensionless groups (numerics)

Ratio of terms in table 3.1	III : I	IV : I	V : I	II : III	IV : II	V : II	IV : III	V : III	IV : V
Mass	$\frac{Dt}{L^2}$	$\frac{rt}{c}$	$\frac{k_m t}{L}$	$\frac{vL}{D}$ $\boxed{\mathrm{Bo}}$	$\frac{rL}{vc}$ $\boxed{\mathrm{Da\,I}}$	$\frac{k_m}{v}$ $\boxed{\mathrm{Me}}$	$\frac{rL^2}{Dc}$ $\boxed{\mathrm{Da\,II}}$	$\frac{k_m L}{D}$ $\boxed{\mathrm{Sh}}$	$\frac{rL}{k_m c}$
Energy	$\frac{\lambda t}{c_p \rho L^2}$ $\boxed{\mathrm{Fo}}$	$\frac{\dot{q} t}{c_p \rho T}$	$\frac{ht}{c_p \rho L}$	$\frac{c_p \rho v L}{\lambda}$ $\boxed{\mathrm{Pe}}$	$\frac{\dot{q} L}{c_p \rho T v}$ $\boxed{\mathrm{Da\,III}}$	$\frac{h}{c_p \rho v}$ $\boxed{\mathrm{St}}$	$\frac{\dot{q} L^2}{\lambda T}$ $\boxed{\mathrm{Da\,IV}}$	$\frac{hL}{\lambda}$ $\boxed{\mathrm{Nu}}$	$\frac{\dot{q} L}{hT}$
Momentum	$\frac{\eta t}{\rho L^2}$	$\frac{ft}{\rho v}$	$\frac{\tau t}{\rho v L}$	$\frac{\rho v L}{\eta}$ $\boxed{\mathrm{Re}}$	$\frac{fL}{\rho v^2}$ $\boxed{\mathrm{We}}$	$\frac{\tau}{\rho v^2}$ $\boxed{\mathrm{Fa}}$	$\frac{fL^2}{\eta v}$ $\boxed{\mathrm{Po}}$	$\frac{\tau L}{\eta v}$ $\boxed{\mathrm{Bm}}$	$\frac{fL}{\tau}$

MEANING OF SYMBOLS

- a = surface per unit of volume
- c = concentration
- c_p = specific heat
- D = diffusivity
- e = electric charge
- E = modulus of elasticity
- f_{el} = electric field per unit of volume
- g = gravitational acceleration
- h = heat transfer coefficient
- k = reaction rate constant
- k_m = mass transfer coefficient
- l = length per unit of volume
- L = characteristic length
- p = pressure
- t = time
- T = temperature
- v = velocity
- x = length coordinate

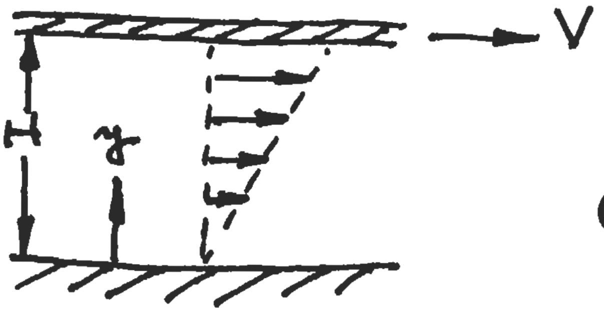
- γ = surface tension
- η = viscosity
- λ = heat conductivity
- ρ = density
- τ = shear stress
- ω = angular frequency

- r = reaction rate per unit of volume
  - first order $r = kc$
  - second order $r = kc^2$ etc.
- $\dot{q}$ = heat production rate per unit of volume
- f = force per unit of volume
  - gravitational $f = g\rho$
  - centrifugal $f = \omega^2 L \rho$
  - pressure gradient $f = \Delta p / L$
  - elastic $f = E/L$
  - surface tension $f = \gamma / L^2$
  - electric $f = e f_{el}$

NUMERICS (see Gen. Ref.)

- Bm = Bingham
- Bo = Bodenstein
- Da = Damköhler
- Fa = Fanning
- Fo = Fourier
- Me = Merkel
- Nu = Nusselt
- Pe = Péclet
- Po = Poiseuille
- Re = Reynolds
- Sh = Sherwood
- St = Stanton
- We = Weber

Couette (drag) flow



$$p = v = \frac{\partial}{\partial x} = 0$$

(simple shearing flow)

$$\rho\left[\cancelto{\text{steady}}{\frac{\partial u}{\partial t}} + \cancelto{0}{u\frac{\partial u}{\partial x}} + \cancelto{0}{v\frac{\partial u}{\partial y}}\right] = -\cancelto{0}{\frac{\partial p}{\partial x}} + \mu\left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{d^2 u}{dy^2} = 0 \rightarrow \frac{du}{dy} = C_1 \rightarrow u(y) = C_1 y + C_2$$

$$u(0) = 0 \rightarrow C_2 = 0\ ,\ u(H) = V \rightarrow C_1 = V/h$$

$$u(y) = \frac{y}{H} V$$

$$\tau_w = \frac{F}{A} = \mu \dot{\gamma}_w = \mu \left(\frac{\partial u}{\partial y}\right)_w = \mu \frac{V}{H}$$

heat generation: $Q = \tau \dot{\gamma} = \mu \dot{\gamma}^2 = \mu \left(\frac{V}{h}\right)^2$

Temperature profile

$$\rho c\left[\cancel{\frac{\partial T}{\partial t}} + \cancel{u\frac{\partial T}{\partial x}} + \cancel{v\frac{\partial T}{\partial y}}\right] = Q + k\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{d^2T}{dy^2} = -\frac{Q}{k} - \frac{\mu}{k}\left(\frac{V}{H}\right)^2 \rightarrow \frac{dT}{dy} = -\frac{Q}{k}y + C_1$$

$$T(y) = -\frac{Q}{k}y^2 + C_1 y + C_2$$

Forced (Dirichlet) b.c.: $T(0) = T_1$, $T(H) = T_2$

Natural (Cauchy) b.c.: $T(0) = T_1$

$$H \quad \frac{\Uparrow h(T_H - T_a)}{\Uparrow -k\nabla T} \rightarrow \frac{dT(y)}{dy} = -h\,(T(H) - T_2)$$

Temperature distribution in drag flow

```
> restart:with(DEtools):
> ode:=diff(T(y),y,y)=-Q/k;
```

$$ode := \frac{\partial^2}{\partial y^2}\mathrm{T}(y) = -\frac{Q}{k}$$

Forced ("Dirichlet") boundary conditions:

```
> T_f:=simplify(dsolve({ode,T(0)=0,T(1)=0},T(y)));
```

$$T\_f := \mathrm{T}(y) = -\frac{1}{2}\frac{Q\,y\,(y-1)}{k}$$

```
> Digits:=4:k:=1:Q:=1:eq1:=rhs(T_f):
```

Natural ("Cauchy") boundary conditions

```
> T_n:=simplify(dsolve({ode,T(0)=0},T(y)));
```

$$T\_n := \mathrm{T}(y) = -\frac{1}{2}y^2 + \_C1\,y$$

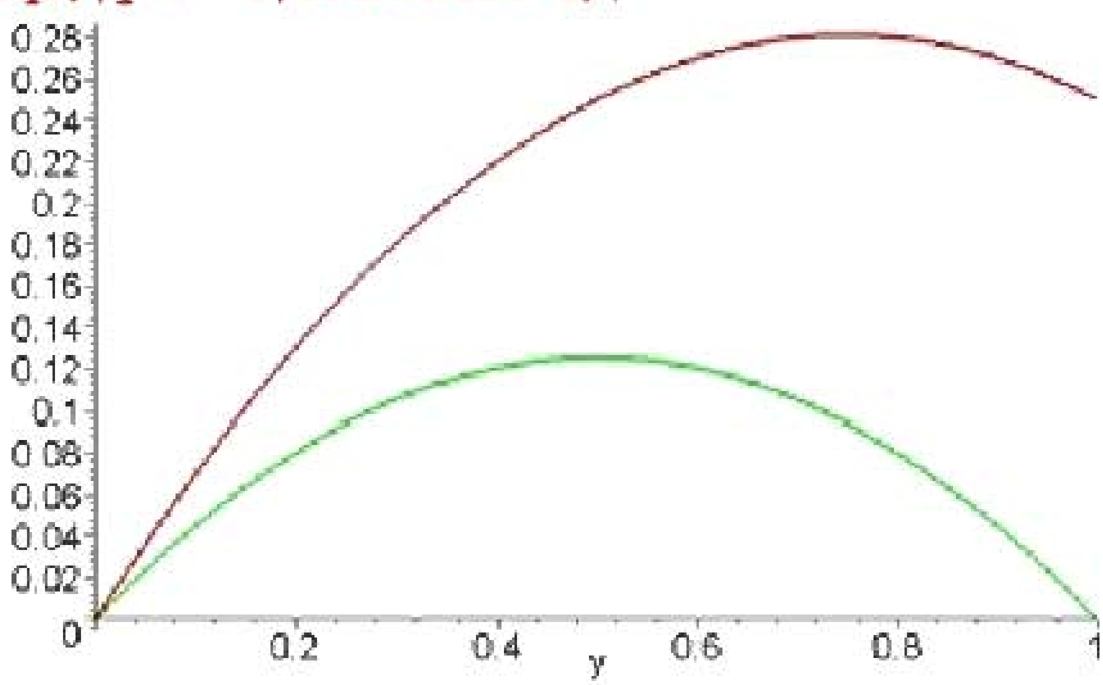
```
> bc_n:=subs(y=1,diff(rhs(T_n),y))=-subs(y=1,rhs(T_n));
```

$$bc\_n := -1 + \_C1 = \frac{1}{2} - \_C1$$

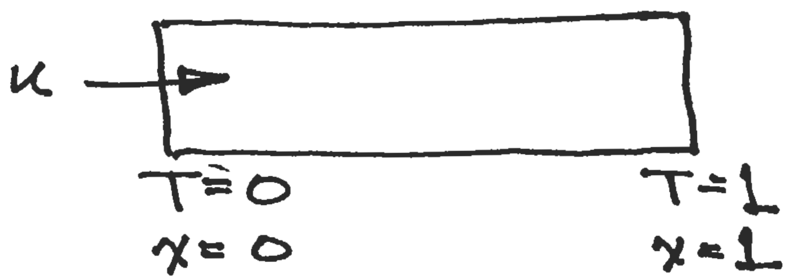
```
> solve(subs(Q=1,bc_n),_C1);
```

$$\frac{3}{4}$$

```
> _C1:=3/4:eq2:=rhs(T_n):
> plot({eq1,eq2},y=0..1,thickness=3);
```



Advective Transport



$$\rho c\left[\cancel{\frac{\partial T}{\partial t}} + u\frac{\partial T}{\partial x} + \cancel{v\frac{\partial T}{\partial y}}\right] = \cancel{Q} + k\left(\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}}\right)$$

$$u\frac{dT}{dx} = \frac{k}{\rho c}\frac{d^2T}{dx^2} \qquad \alpha = \frac{k}{\rho c}$$

$$Pe\frac{dT}{dx} = \frac{d^2T}{dx^2}, \qquad Pe = \frac{uL}{\alpha}$$

1-D heat transport by diffusion and advection

```
> restart:with(DEtools):
> ode:= Pe*diff(T(x),x)=diff(T(x),x,x);
```

$$ode := Pe\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) = \frac{\partial^2}{\partial x^2}\mathrm{T}(x)$$

```
> TT:=simplify(dsolve({ode,T(0)=0,T(1)=1},T(x)));
```

$$TT := \mathrm{T}(x) = \frac{-1 + \mathbf{e}^{(Pe\, x)}}{-1 + \mathbf{e}^{Pe}}$$

```
> eq1:=subs(Pe=1,rhs(TT)):eq5:=subs(Pe=5,rhs(TT)):eq10:=subs(Pe=10,r
  hs(TT)):
> plot({eq1,eq5,eq10},x=0..1,thickness=3);
```