# 3.044 MATERIALS PROCESSING

LECTURE 4

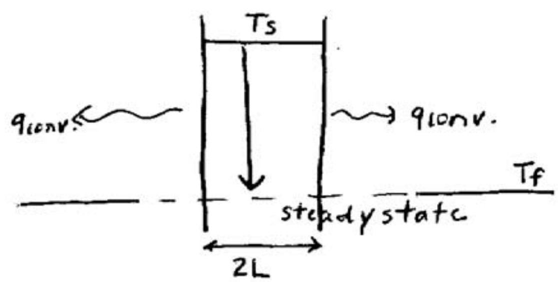
**General Heat Conduction Solutions:**

$\frac{\partial T}{\partial t} = \nabla \cdot k\nabla T,\ T(\bar{x}, t)$

Trick one: steady state $\nabla^2 T = 0,\ T(x)$

Trick two: low Biot number $\frac{\partial T}{\partial t} = \alpha\, h(T_s - T_f),\ T(t)$

**Low Biot Number Solutions:** Newtonian Heating / Cooling




$\frac{hL}{K} < 0.1$

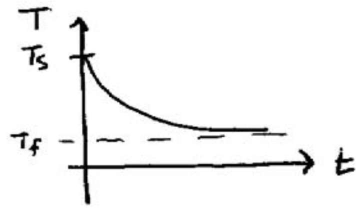
Global Heat Balance:

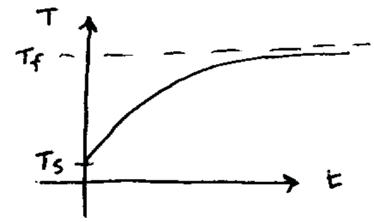
$$
\begin{aligned}
q_{\text{conv}} &= q_{\text{lost}} \\
A\, h(T - T_f) &= -\rho\, c_p \frac{\partial T}{\partial t} V \\
\int \frac{\partial T}{T - T_f} &= \int \frac{-hA}{\rho c_p V} \partial t \\
\ln(T - T_f) &= \frac{-hA}{\rho\, c_p\, V} t + C \\
@t = 0, T &= T_s \\
\ln(T_s - T_f) &= C
\end{aligned}
$$

*Date*: February 21st, 2012.

$$\ln\left(\frac{T-T_f}{T_s-T_f}\right) = \frac{-hA}{\rho\, c_p\, V}\, t$$

$$\boxed{\frac{T-T_f}{T_s-T_f} = e^{\frac{-hA}{\rho\, c_p\, V}\, t}}$$




<u>Transient Heat Conduction:</u> depends on position and time

$$\boxed{\frac{\partial T}{\partial t} = \alpha\, \nabla^2\, T}$$

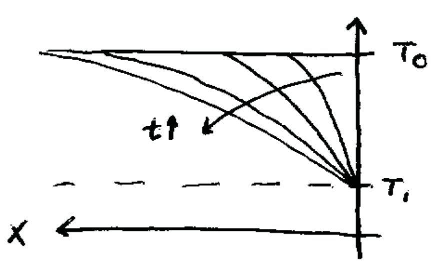
You should **know**:

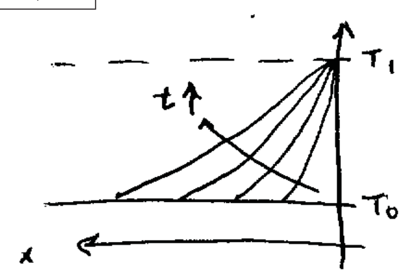
1) Some common solutions for simple geometries
2) Where to find solutions
3) How to build up complex solutions using simple solutions

**Semi-Infinite Solid**

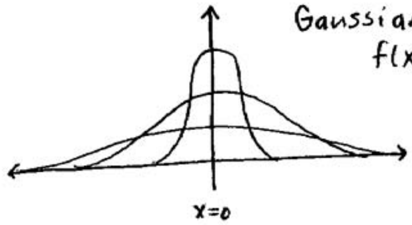
- constant $T_1$ at surface
- initially $T_0$ everywhere

$$\boxed{\frac{T-T_0}{T_1-T_0} = \text{erfc}\frac{x}{2\sqrt{\alpha t}}}$$

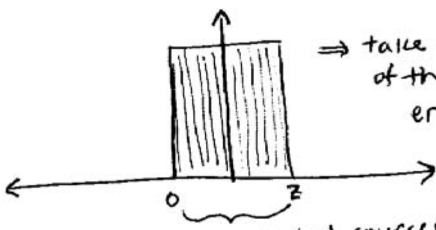


$$\boxed{\operatorname{erf}(z) = \int_0^z e^{-x^2}\,dx}$$



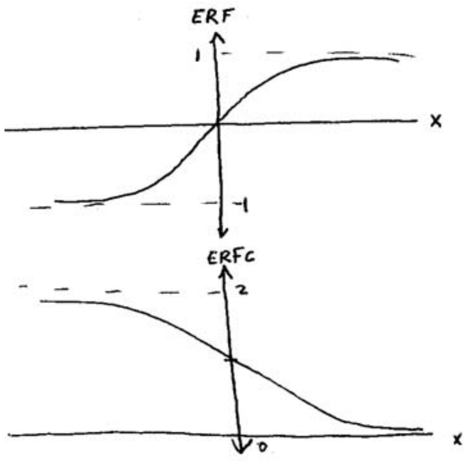

Gaussian Distribution
$f(x) = e^{-x^2}$




$\Rightarrow$ take the integral of the gaussians: erf

many point sources, each with a gaussian

$$\boxed{\operatorname{erfc} = 1 - \operatorname{erf}}$$




$$\operatorname{erf}(x) = \begin{cases} -1 < x < 1, & \operatorname{erf} \cong x \\ x > 2, & \cong 1 \\ x < -2, & \cong -1 \end{cases}$$

$$T(x) = (T_1 - T_0)\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}}\right) + T_0$$

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}}\right)$$

$$(-1)\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}}\right)(-1)$$

$$\frac{T_0 - T}{T_1 - T_0} = \text{erf}\left(\frac{x}{2\sqrt{\alpha\, t}}\right) - (T - T_0)$$
$$\frac{T - T_1}{T_1 - T_0} = -\text{erf}\left(\frac{x}{2\sqrt{\alpha\, t}}\right)$$
$$\frac{T - T_1}{T_0 - T_1} = \text{erf}\left(\frac{x}{2\sqrt{\alpha\, t}}\right)$$

**Semi-Infinite Solid**

- convection at surface: $q_{\text{lost}} = h(T - T_f)$

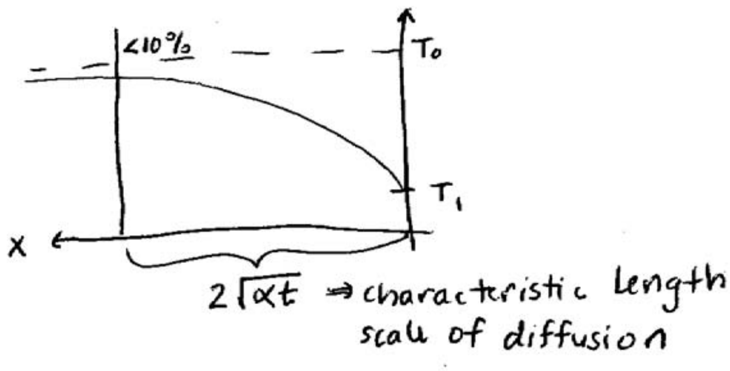
$$\boxed{\Theta = ERFC\left(\frac{x}{2\sqrt{\alpha\, t}}\right) - EXP\left(\frac{hx}{k} + h^2 k\, \alpha\, t\right) \cdot ERFC\left(\frac{x}{2\sqrt{\alpha\, t}} + \frac{h}{k}\sqrt{\alpha\, t}\right)}$$

Where to find these solutions:

- Carslaw & Jaeger
- Crank

**Dimensionless Numbers:**

$$\frac{T - T_0}{T_1 - T_0} = \text{erfc}\frac{x}{2\sqrt{\alpha t}}$$
$$\frac{T - T_0}{T_1 - T_0} = \Theta$$
$$\chi = \frac{x}{L}$$
$$x = L\chi$$
$$\tau = \frac{\alpha\, t}{L^2}$$
$$t = \frac{L^2 T}{\alpha}$$
$$\Theta = \text{erfc}\left(\frac{L\chi}{2\sqrt{\frac{\alpha\, L^2\, \tau}{\alpha}}}\right)$$
$$\boxed{\Theta = \text{erfc}\left(\frac{\chi}{2\sqrt{\tau}}\right)}$$



<10%
$T_0$
$T_1$
x
$2\sqrt{\alpha t}$ ⇒ characteristic length scale of diffusion