Lecture 22
Electromagnetic Waves

Program:

1. Energy carried by the wave (Poynting vector).
2. Maxwell’s equations and Boundary conditions at interfaces.
3. Materials boundaries: reflection and refraction. Snell’s Law.

Questions you should be able to answer by the end of today’s lecture:

1. What is the direction of energy flux of the EM wave?
2. What is the basic principle behind the boundary conditions for EM waves?
3. Continuity of what wave parameter is responsible for reflection and Snell’s laws?

<u>**Reflection and Refraction at Materials Interfaces**</u>

Boundary Conditions: Continuity conditions for the fields obeying Maxwell's Equations. These conditions can be derived from application of Maxwell's equations, Gauss and Stokes Theorems and have to be satisfied at any materials boundary.

1. $\hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=0 \Rightarrow B_{1\perp}=B_{2\perp}$

The component of the magnetic induction perpendicular (normal) to the interface is continuous across the interface.

2. $\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\sigma \;\Rightarrow D_{2\perp}-D_{1\perp}=\sigma$ ($\sigma=$surface charge density)

In the presence of surface charge at the material interface, the normal component of the electric displacement changes abruptly by an amount equal to surface charge density σ.

3. $\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=0 \Rightarrow \vec{E}_{1\|}=\vec{E}_{2\|}$

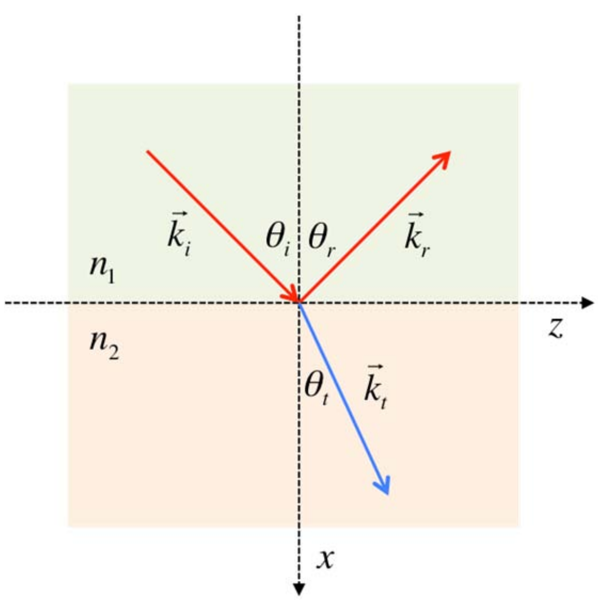
The component of the electric field parallel (tangential) to the interface is continuous across the interface.

4. $\hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{K} \Rightarrow \vec{H}_{2\|}-\vec{H}_{1\|}=\vec{K}$ ($\vec{K}=$surface current density)

In the presence of a surface current at the interface, the component of the magnetic induction parallel (tangential) to the interface changes abruptly by the amount equal to surface current $\vec{K}$.

In many cases in optics, the surface charge density and surface current density are zero, and consequently the normal components of $\vec{D}$ and $\vec{B}$ and the tangential components of $\vec{E}$ and $\vec{H}$ are continuous.

Consider a charge-free current-free interface between materials with refractive indices $n_1,\ n_2$.



Consider a situation where a wave is incident form the top onto the interface

$\vec{E}_i e^{i\vec{k}_i\cdot\vec{r}-i\omega t}$ incident wave

$\vec{E}_r e^{i\vec{k}_r\cdot\vec{r}-i\omega t}$ reflected wave

$\vec{E}_t e^{i\vec{k}_t\cdot\vec{r}-\omega t}$ transmitted wave

The dispersion relations for a homogeneous medium tell us:

$$\left|\vec{k}_i\right|=\left|\vec{k}_r\right|=\frac{\omega n_1}{c_0}$$

$$\left|\vec{k}_t\right|=\frac{\omega n_2}{c_0}$$

Due to phase continuity, the phases of all three waves (incident, reflected and transmitted) have to be equal at the interface plane $x = 0$. Consequently the phases of the reflected and transmitted waves are completely determined by the phase of the incident wave.

$$\left(\vec{k}_i \cdot \vec{r}\right)_{x=0} = \left(\vec{k}_r \cdot \vec{r}\right)_{x=0} = \left(\vec{k}_t \cdot \vec{r}\right)_{x=0}$$

$$\left(k_{iy} y + k_{iz} z\right) = \left(k_{ry} y + k_{rz} z\right) = \left(k_{ty} y + k_{tz} z\right) \Rightarrow \begin{cases} k_{iy} = k_{ry} = k_{ty} \\ k_{iz} = k_{rz} = k_{tz} \end{cases}$$

We were able to make these conclusions because z and y are arbitrary coordinates.

The equations above have two important consequences:

1. The vectors $\vec{k}_i$, $\vec{k}_r$, $\vec{k}_t$ all lie in a plane called *the plane of incidence*. We have oriented our coordinate system such that the plane of incidence coincides with the x-z plane. Then electric field can written in the following form: $\vec{E} = \vec{E}_0 e^{i\left(k_x x + k_z z - \omega t\right)}$
2. The tangential components of the wavevector (components lying within *the plane of incidence*) are identical regardless of the medium that they are in: $k_{iz} = k_{rz} = k_{tz} \equiv \beta$.

   Then:

$$\left|\vec{k}_i\right| = \left|\vec{k}_r\right| = \frac{\omega n_1}{c_0},\ k_{iz} = \frac{\omega n_1}{c_0}\sin\theta_i \quad and \quad k_{rz} = \frac{\omega n_1}{c_0}\sin\theta_r$$

$$\left|\vec{k}_t\right| = \frac{\omega n_2}{c_0},\ k_{tz} = \frac{\omega n_2}{c_0}\sin\theta_t$$

$$k_{iz} = k_{rz} = k_{tz}$$

From these equations we find:

1. **Angle of reflection equals angle of incidence:**

   $$k_{iz} = k_{rz} \Rightarrow \sin\theta_i = \sin\theta_r \Rightarrow \theta_i = \theta_r$$

2. **Snell's law:**

   $$\left.\begin{aligned} k_{iz} &= n_1 \frac{\omega}{c}\sin\theta_i \\ k_{tz} &= n_2 \frac{\omega}{c}\sin\theta_t \end{aligned}\right\} \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$$

Total internal reflection. Waveguides.

As you remember from the Snell's law:

$$n_1 \sin\theta_i = n_2 \sin\theta_t \Rightarrow \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$$

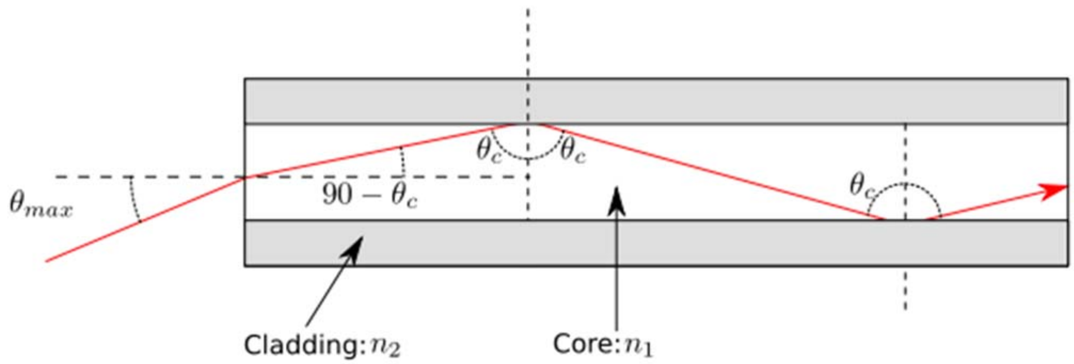
This implies that if $n_1 < n_2$ then $\theta_t < \theta_i$ and if $n_1 > n_2$ then $\theta_t > \theta_i$.

Which means that in the case of $n_1 > n_2$ for a certain angle of incidence $\theta_i = \theta_c$ the refraction angle θ_t becomes equal to 90°, which in practice means that the light cannot escape through the interface and will stay inside the material with higher refractive index.

This effect is called total internal reflection, and the critical angle is simply:

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}\sin\theta_t = \frac{n_2}{n_1}}$$

Optical fibers and waveguides used for transmission of information over the long distances use this principle to keep the EM waves inside. Waveguides consist of higher refractive index (n_1) **core** and lower refractive index (n_2) **cladding**:



Then we can find the maximum angle at which we can still "couple" EM waves into the waveguide so we can take advantage of the total internal reflection.

The refractive index for air is $n = 1$, then we can find:

$$\left.\begin{array}{c} \sin\theta_c = \dfrac{n_2}{n_1} \\ 1\cdot\sin\theta_{\max} = n_1\sin\left(\dfrac{\pi}{2}-\theta_c\right) = n_1\cos\theta_c \end{array}\right\} \Rightarrow \sin\theta_{\max} = n_1\cos\theta_c = n_1\sqrt{1-\sin^2\theta_c} = n_1\sqrt{1-\left(\frac{n_2}{n_1}\right)^2}$$

$$\sin\theta_{\max} = \sqrt{n_1^2 - n_2^2}$$

Another important characteristic connected to the maximum angle is numerical aperture (NA), and in case of coupling between air ($n = 1$) and fiber:

$$\boxed{NA = 1\cdot\sin\theta_{\max} = \sqrt{n_1^2 - n_2^2}}$$

s-p Polarization: transmission and reflection coefficients

In general we express the electric fields on both sides of the interface as:

$$\vec{E} = \begin{cases} \left(\vec{E}_i e^{i\vec{k}_i\cdot\vec{r}} + \vec{E}_r e^{i\vec{k}_r\cdot\vec{r}}\right)e^{-i\omega t} & x<0 \\ \vec{E}_t e^{i\vec{k}_t\cdot\vec{r}} e^{-i\omega t} & x>0 \end{cases}$$

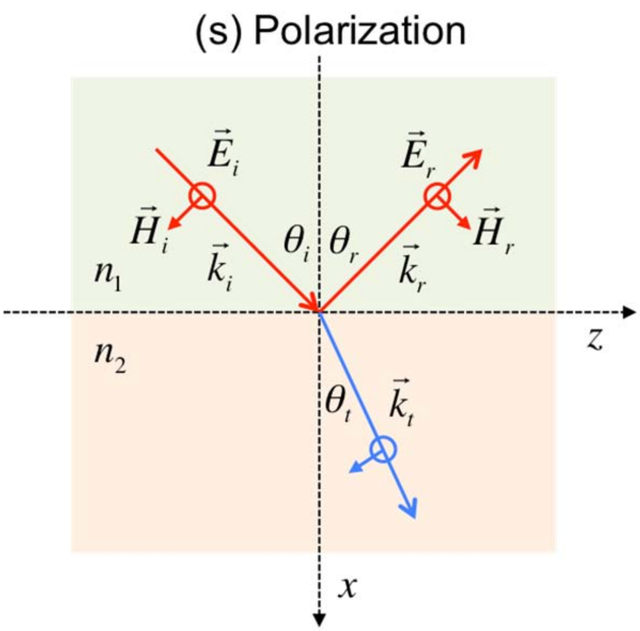
It is useful to separate the electric field into components. These components are orthogonal to each other and are called Polarizations. Every wave can be represented as a superposition of 2 polarizations.

- (p) In the plane of incidence ($x-z$ plane)
- (s) Perpendicular to the plane of incidence (y-direction).

Let's consider both polarizations separately.

s-Polarized electric field:

$$\vec{E}_s\left(x,y,z\right)=\left(0,E_s\left(x,y,z\right),0\right)$$



The electric field above the interface in the material n_1 will be equal to:

$$\vec{E}_s = \hat{y}\left(\underbrace{E_{is}e^{ik_{ix}x}}_{incident} + \underbrace{E_{rs}e^{-ik_{rx}x}}_{reflected}\right)e^{i\omega t+i\beta z}$$

This component is tangential (parallel) to the material interface and hence, according to the boundary condition $\vec{E}_{1\|} = \vec{E}_{2\|}$, will be continuous across the interface: $E_{iy}+E_{ry}=E_{ty}$.

Recall that:

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \vec{\nabla}\times\vec{E}+\mu\frac{\partial\vec{H}}{\partial t}=0$$

Substituting the plane waveform: $\vec{H}\left(x,y,z\right)=\vec{H}_0 e^{i\vec{k}\vec{r}-i\omega t}$ into the equation above, we find:

$$\vec{\nabla}\times\vec{E}+i\omega\mu\vec{H}=0$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

Since $\vec{E} \perp \vec{H}$, and the electric field is perpendicular to the plane of incidence, the magnetic field will lie in the plane of incidence and perpendicular to the wavevector. Recall the boundary condition for the tangential component (parallel to the interface, z in our case) of the magnetic field: The tangential component of the magnetic field $H_{2\parallel} - H_{1\parallel} = J$.

In the absence of currents ($H_{1\parallel} = H_{2\parallel}$) the condition above implies continuity of the z-component of the magnetic field across the interface (in $x = 0$ plane): $H_{iz} + H_{rz} = H_{tz}$

$$H_{iz} = \frac{1}{i\omega\mu_0}\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right) = \frac{1}{i\omega\mu_0}\frac{\partial}{\partial x}E_i e^{i\left(k_{ix}x + k_{iz}z\right)} = ik_{ix}\frac{1}{i\omega\mu_0}E_i e^{i\left(k_{ix}x + k_{iz}z\right)}$$

Recall: $k_{ix} = \left|\vec{k}_i\right|\cos\theta_i = \frac{n_1\omega}{c_0}\cos\theta_i \quad and \quad k_{tx} = \left|\vec{k}_t\right|\cos\theta_t = \frac{n_2\omega}{c_0}\cos\theta_t$

$$H_{iz} = \frac{n_1}{c_0\mu_0}\cos\theta_i E_i e^{i\left(k_{ix}x + k_{iz}z\right)} = \frac{n_1}{c_0\mu_0}\cos\theta_i E_i e^{i\left(k_{ix}x + \beta z\right)}$$

$$H_{rz} = -\frac{n_1}{c_0\mu_0}\cos\theta_i E_r e^{i\left(-k_{ix}x + k_{iz}z\right)} = -\frac{n_1}{c_0\mu_0}\cos\theta_i E_r e^{i\left(k_{ix}x + \beta z\right)}$$

$$H_{tz} = -\frac{n_t}{c_0\mu_0}\cos\theta_t E_t e^{i\left(-k_{tx}x + k_{tz}z\right)} = \frac{n_2}{c_0\mu_0}\cos\theta_t E_t e^{i\left(k_{tx}x + \beta z\right)}$$

Consequently the boundary conditions will lead us to two equations for electric field amplitudes for the reflected and transmitted waves:

$$\left.\begin{array}{r} E_{iy} + E_{ry} = E_{ty} \\ H_{iz} + H_{rz} = H_{tz} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} E_{iy} + E_{ry} = E_{ty} \\ n_1\cos\theta_i E_{iy} - n_1\cos\theta_i E_{ry} = n_2\cos\theta_t E_{ty} \end{array}\right.$$

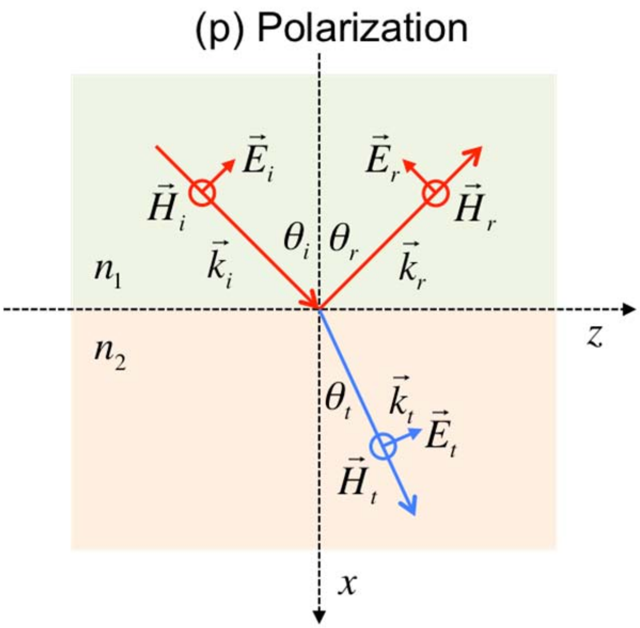
Then we can find reflection and transmission coefficients for the s-polarized wave:

$$r_s = \left(\frac{E_{ry}}{E_{iy}}\right) = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$$

$$t_s = \left(\frac{E_{ty}}{E_{iy}}\right) = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}$$

Then the reflectivity of the interface is simply: $R = \left|r_s\right|^2$

p-Polarization:



$$\vec{E}_p(x,y,z) = \left(E_{px}(x,y,z), 0, E_{pz}(x,y,z)\right)$$

In the absence of interface charge the boundary condition electrical fields perpendicular to the material interface is:

$$D_{1\perp} = D_{2\perp} \Rightarrow \varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$$

This implies that the z-component of electric field has to be conserved across the interface:

$$\varepsilon_1 E_{iz} + \varepsilon_1 E_{rz} = \varepsilon_2 E_{tz}$$

We also know that the components of the electrical field parallel to the interface have to be conserved across the interface: $\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$

Which in this case means that x-component of the electric field have to be conserved across the interface:

$$E_{ix} + E_{rx} = E_{tx}$$

Combining the equations we got from the boundary conditions we get:

$$\left.\begin{array}{c} \varepsilon_1 E_{iz} + \varepsilon_1 E_{rz} = \varepsilon_2 E_{tz} \Rightarrow \varepsilon_1 \sin\theta_i E_i + \varepsilon_1 \sin\theta_i E_r = \varepsilon_2 \sin\theta_t E_t \\ n_1 \sin\theta_i = n_2 \sin\theta_t \quad and \quad \varepsilon_1 = n_1^2, \ \varepsilon_2 = n_2^2 \end{array}\right\} \Rightarrow n_1 E_i + n_1 E_r = n_2 E_t$$

$$E_{ix} + E_{rx} = E_{tx} \Rightarrow \cos\theta_i (E_i - E_r) = \cos\theta_t E_t$$

Leading to the transmission and reflection coefficients:

$$r_p = \left(\frac{E_r}{E_i}\right) = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

$$t_p = \left(\frac{E_t}{E_i}\right) = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

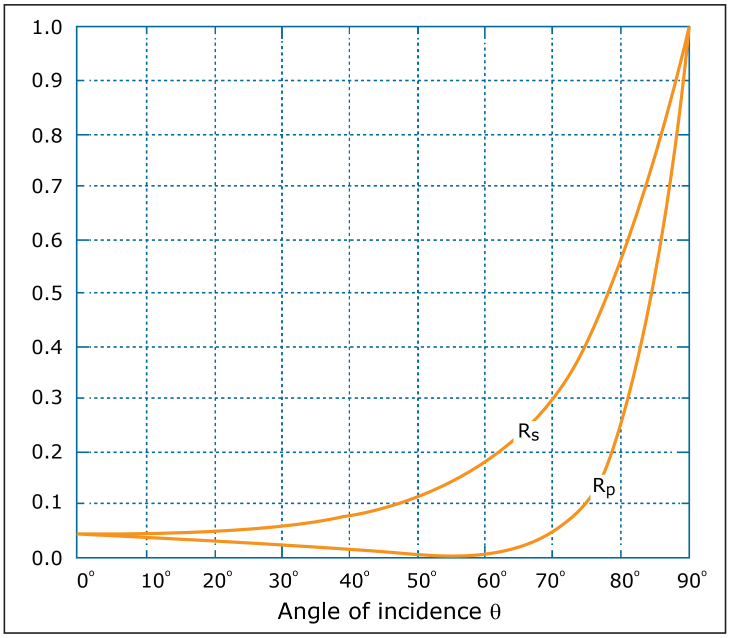
Note that for p-polarization there exists an angle known as Brewster angle at which the reflection coefficient is zero and all the light is transmitted:

$$r_p = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = 0 \Rightarrow n_2\cos\theta_i = n_1\cos\theta_t \Rightarrow n_2^2\cos^2\theta_i = n_1^2\cos^2\theta_t$$

$$\Rightarrow n_2^2\left(1-\sin^2\theta_i\right) = n_1^2\left(1-\sin^2\theta_t\right) \Rightarrow n_2^2\left(1-\sin^2\theta_i\right) = n_1^2\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_i\right)$$

$$\left(n_1^4 - n_2^4\right)\sin^2\theta_i = n_2^2\left(n_1^2 - n_2^2\right)$$

$$\sin^2\theta_i = \frac{n_2^2}{n_1^2 + n_2^2}$$



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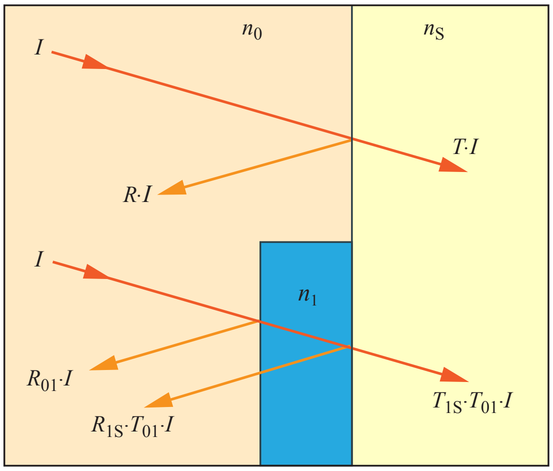
Anti-reflection coatings: maximizing the coupling of light into the material.

Many applications such as solar panels, optical interconnects require maximum coupling of the incoming light into the material. As you have seen above at intersection of any two materials with different refractive indices there is significant reflection, which is highly undesirable for the above-mentioned applications. It is possible, however to create a coating at the interface of the two materials that would minimize the reflection between them.



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By using the reflection and transmission coefficients at both surfaces one can find that the reflection coefficient will be minimal when:

$$n_I = \sqrt{n_0 n_S}$$

Alternatively one can use quarter-wave coatings. These coatings are precisely $d = \frac{\lambda_0}{4n_I}$ thick and they work by making the waves reflected from the first and second interfaces be exactly out of phase and hence annihilate each other.

However, note that quarter-wave coatings work best for a particular wavelength, which they have been designed for but matching indices of refraction is general solution that works for most materials with low dispersion.

3.024 Electronic, Optical and Magnetic Properties of Materials
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