Oct. 03 2005: Lecture 8:

Complex Numbers and Euler's Formula

Reading:
Kreyszig Sections: §12.1 (pp:652–56) , §12.2 (pp:657–62) , §12.6 (pp:679–82) , §12.7 (pp:682–85)

Complex Numbers and Operations in the Complex Plane

With $\imath \equiv \sqrt{-1}$, the complex numbers can be defined as the space of numbers spanned by the vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \imath \end{pmatrix} \tag{8-1}$$

so that any complex number can be written as

$$z = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ \imath \end{pmatrix} \tag{8-2}$$

or just simply as

$$z = x + iy \tag{8-3}$$

where x and y are real numbers. $\mathrm{Re}z \equiv x$ and $\mathrm{Im}z \equiv y$.

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Operations on complex numbers

Addition, subtraction, multiplication, division

Complex Plane and Complex Conjugates . . . .

Because the complex basis can be written in terms of the vectors in Equation 8-1, it is natural to plot complex numbers in two dimensions—typically these two dimensions are the "complex plane" with $(0, \imath)$ associated with the y-axis and $(1, 0)$ associated with the x-axis.

The reflection of a complex number across the real axis is a useful operation. The image of a reflection across the real axis has some useful qualities and is given a special name—"the complex conjugate."

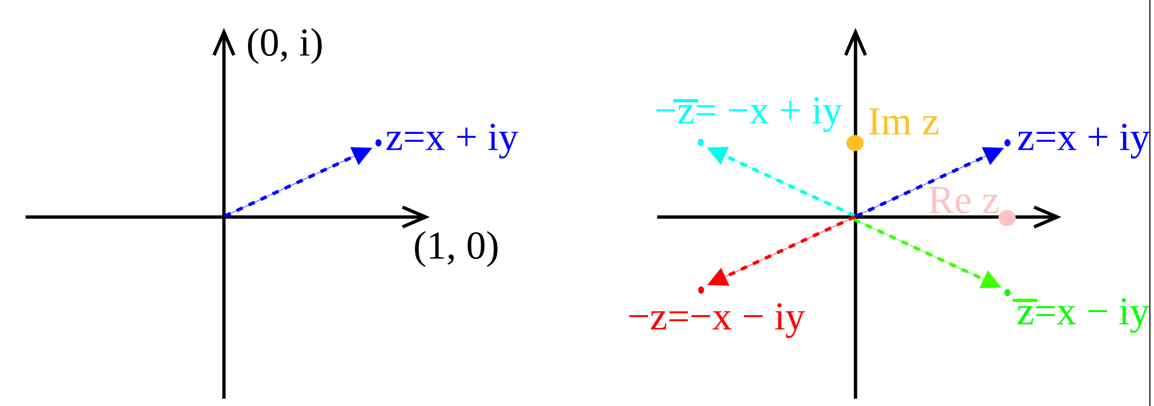

Figure 8-1: Plotting the complex number z in the complex plane: The complex conjugate $(\bar{z})$ is a *reflection* across the real axis; the minus $(-z)$ operation is an *inversion* through the origin; therefore $-(\bar{z}) = (\overline{-z})$ is equivalent to either a reflection across the imaginary axis or an inversion followed by a reflection across the real axis.
The real part of a complex number is the projection of the displacement in the real direction and also the average of the complex number and its conjugate: Re$z = (z+\bar{z})/2$. The imaginary part is the displacement projected onto the imaginary axis, or the complex average of the complex number and its reflection across the imaginary axis: Im$z = (z - \bar{z})/(2\imath)$.

Polar Form of Complex Numbers

There are physical situations in which a transformation from Cartesian (x, y) coordinates to polar (*or cylindrical*) coordinates (r, θ) simplifies the algebra that is used to describe the physical problem.

An equivalent coordinate transformation for complex numbers, $z = x + \imath y$, has an analogous simplifying effect for *multiplicative operations* on complex numbers. It has been demonstrated how the complex conjugate, $\bar{z}$, is related to a reflection—multiplication is related to a **counter-clockwise** rotation in the complex plane. Counter-clockwise rotation corresponds to increasing θ.

The transformations are:

$$
\begin{aligned}
(x, y) \rightarrow (r, \theta) & \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \\
(r, \theta) \rightarrow (x, y) & \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}
\end{aligned}
\tag{8-4}
$$

where $\arctan \in (-\pi, \pi]$.

Multiplication, Division, and Roots in Polar Form ....

One advantage of the polar complex form is the simplicity of multiplication operations:

DeMoivre's formula:

$$z^n = r^n(\cos n\theta + \imath \sin n\theta) \tag{8-5}$$

$$\sqrt[n]{z} = \sqrt[n]{z}(\cos \frac{\theta + 2k\pi}{n} + \imath \sin \frac{\theta + 2k\pi}{n}) \tag{8-6}$$

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Polar Form of Complex Numbers *Writing a function to convert to polar form*

Exponentiation and Relations to Trignometric Functions

Exponentiation of a complex number is defined by:

$$e^z = e^{x+\imath y} = e^x(\cos y + \imath \sin y) \tag{8-7}$$

Exponentiation of a purely imaginary number advances the angle by rotation:

$$e^{\imath y} = \cos y + \imath \sin y \tag{8-8}$$

combining Eq. 8-8 with Eq. 8-7 gives the particularly useful form:

$$z = x + \imath y = r e^{\imath \theta} \tag{8-9}$$

and the useful relations (that can be obtained simply by considering the geometry of the complex plane)

$$e^{2\pi\imath} = 1 \quad e^{\pi\imath} = -1 \quad e^{-\pi\imath} = -1 \quad e^{\frac{\pi}{2}\imath} = \imath \quad e^{-\frac{\pi}{2}\imath} = -\imath \tag{8-10}$$

Judicious subtraction of powers in Eq. 8-8 and generalization gives the following useful relations for trigonometric functions:

$$\begin{aligned} \cos z = \frac{e^{\imath z} + e^{-\imath z}}{2} \qquad & \sin z = \frac{e^{\imath z} - e^{-iz}}{2\imath} \\ \cosh z = \frac{e^{z} + e^{-z}}{2} \qquad & \sinh z = \frac{e^{z} - e^{-z}}{2} \\ \cos z = \cosh \imath z \qquad & \imath \sin z = \sinh \imath z \\ \cos \imath z = \cosh z \qquad & \sin \imath z = \imath \sinh z \end{aligned} \tag{8-11}$$

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Numerical precision and rounding of complex numbers

Numerical and symbolic representations of complex numbes

Roots of polynomial equations

Handling complex roots of polynomial equations