Extension Theorem

   We'll now look at a couple of theorems which, while not deep or surprising, are worth noticing because they're useful as a tool for proving other theorems. The first doesn't involve any semantics at all. It only depends on the grammatical fact that a sentence is built up out of atomic sentences in stages.

> **Simplest Example Lemma.** If there is an SC sentence with a given property, then there is a simplest SC sentence with the property. More precisely, there is an SC sentence having the property such that no proper subsentence has the property.

(A *subsentence* of a given sentence is a sentence that's a part of the given sentence, where we count a sentence as a part of itself. A *proper subsentence* of a given sentence is a subsentence of the given sentence other than itself.)

   The Simplest Example Lemma is perhaps obvious even without proof. If we have a sentence with property _, that sentence was built up in stages, and at the first stage of the construction at which we get a sentence with property _, we'll have our simplest subsentence with property _. Even though it's not that hard to convince oneself of the Simplest Example Lemma even without a detailed proof, I would like to give a proof, because we'll be using the lemma time and again, usually without mentioning it by name. If fact, I'll give two proofs.

First Proof: Start with a sentence χ with the property _, and apply the following program:

1.   START, with the box empty.
2.   Put χ into the box.
3.   Is there any proper subsentence of the sentence currently in the box that has property _. If there is, go to step 6. Otherwise, go on the step 4.
4.   The sentence currently in the box is a simplest sentence with property _, as desired.
5.   STOP.
6.   Find a proper subsentence of the sentence currently in box that has property _ and put in into the box, replacing the sentence that's there already.
7.   Go to step 3.

The sentences that get put in the box at different stages of this procedure all lie along a single branch of the structure tree for χ. Each time we cycle through step 3, we move to a new node farther down the tree. Since every branch of the tree is finite, the process has to halt eventually.<u>X</u>

Second Proof: Recall that the SC sentence make up the smallest collection Ω with the following six properties:

①   Every atomic sentence is in Ω.
②   If φ and ψ are in Ω, so is $(\varphi \vee \psi)$.

③   If φ and ψ are in Ω, so is $(\varphi \wedge \psi)$.
④   If φ and ψ are in Ω, so is $(\varphi \rightarrow \psi)$.
⑤   If φ and ψ are in Ω, so is $(\varphi \leftrightarrow \psi)$.
⑥   If φ is in Ω, so is $\neg \varphi$.

By "smallest" here, what I mean is that Ω has properties ① through ⑥, and that any other set that has properties ① through ⑥ includes Ω.

Let's assume that there isn't a simplest sentence that has the property $\wp$. We intend to show that there isn't any sentence that has property $\wp$. Thus we are proving the Simplest Example Lemma by proving it's *contrapositive*:

> If there isn't a simplest SC sentence with property $\wp$, then there isn't any SC sentence with property $\wp$.

Let Λ be the set of all SC sentences χ such that no subsentence of χ has property $\wp$. Since every sentence is a subsentence of itself, none of the sentences in Λ have property $\wp$. Thus, if we can show that every SC sentence is in Λ, this will suffice to show that no SC sentence has property $\wp$.

We want to show that Λ = {SC sentences}. It's immediate from the definition of "Λ" that $\Lambda \subseteq$ {SC sentences}, so what we need to show is that {SC sentences} $\subseteq \Lambda$. For this, it will suffice to show that Λ satisfies conditions ① through ⑥, since {SC sentences} was defined to be the *smallest* collection satisfying ① through ⑥.

Λ satisfies condition ①: Let φ be an atomic sentence. Atomic sentences don't have any proper subsentences, so, if φ had property $\wp$, it would be a simplest SC sentence with property $\wp$. By hypothesis, there is no simplest sentence with property $\wp$. Consequently, φ does not have property $\wp$. Since the only subsentence of φ is φ itself, it follows that no subsentence of φ has property $\wp$, that is, that φ is in Λ.

Λ satisfies condition ②: Suppose that φ and ψ are in Λ. We want to show that no subsentence of $(\varphi \vee \psi)$ has property $\wp$. The only subsentence of $(\varphi \vee \psi)$ other than $(\varphi \vee \psi)$ itself are the subsentences of φ and the subsentences of ψ. Because φ is in Λ, no subsentence of φ has property $\wp$. Because ψ is in Λ, no subsentence of ψ has property $\wp$. Thus, if we can show that $(\varphi \vee \psi)$ lacks property $\wp$, it will follow that no subsentence of $(\varphi \vee \psi)$ has property $\wp$, and hence that $(\varphi \vee \psi)$ is in Λ, as required. Well, if $(\varphi \vee \psi)$ had property $\wp$, it would be a simplest sentence with property $\wp$, and, by hypothesis, there is no simplest sentence with property $\wp$.

Λ satisfies conditions ③ through ⑤: Similar.

Λ satisfies condition ⑥: Suppose that φ is in Λ. We want to show that no subsentence of $\neg \varphi$ has property $\wp$. The only subsentences of $\neg \varphi$ other than $\neg \varphi$ itself are the subsentences of φ. Because φ is in Λ, no subsentence of φ has property $\wp$. Thus, if we

can show that ¬ φ lacks property $\wp$, it will follow that no subsentence of ¬ φ has property $\wp$, and hence that ¬ φ is in Λ, as required. Well, if ¬ φ had property $\wp$, it would be a simplest sentence with property $\wp$, and, by hypothesis, there is no simplest sentence with property $\wp$.<u>X</u>

Now, let's turn to our other theorem. The semantics we've been developing for the sentential calculus is *truth-functional,* meaning that the truth or falsity of a complex sentence is determined by the truth or falsity of its atomic parts. Consequently, when we determine which atomic sentences are true, we determine the truth or falsity of every sentence of the language. So the following result will come as no surprise:

Extension Theorem. Any function assigning a numerical value, either 0 or 1, to every atomic sentence can be extended to a normal truth assignment in a unique way.

Once again, I'll give two proofs. (I promise not to keep doing this.)

First Proof: There are two parts to the theorem, existence and uniqueness. For each part, we make use of the notion of the *degree* of a sentence, defined as follows: An atomic sentence has degree 0. A sentence of one of the forms $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, or $(\varphi \leftrightarrow \psi)$ has degree one greater than the maximum of the degree of φ and the degree of ψ. A negation has degree one greater than the degree of its negatum.

The degree measures the complexity of a sentence. A compound sentence is more complex — has higher degree than — any of its proper parts.

Existence. Given a function H, assigning a value, 0 or 1, to each atomic sentence, define a sequence $\Im_0, \Im_1, \Im_2, \Im_3,...$ of functions, as follows: $\Im_0$ is just H. Given a function $\Im_n$, assigning a value, either 0 or 1, to the sentences of degree less than or equal to n, define the function $\Im_{n+1}$, assigning a value, either 0 or 1, to the sentences of degree less than or equal to n+1, as follows: If φ has degree less than or equal to n, $\Im_{n+1}(\varphi) = \Im_n(\varphi)$. If φ has degree n+1, then it's either a disjunction, a conjunction, a conditional, a biconditional, or a negation. If φ is a disjunction, say $(\psi \vee \theta)$, we'll set $\Im_{n+1}(\varphi)$ equal to the maximum of $\Im_n(\psi)$ and $\Im_n(\theta)$. If φ is a conjunction, say $(\psi \wedge \theta)$, we'll set $\Im_{n+1}(\varphi)$ equal to the minimum of $\Im_n(\psi)$ and $\Im_n(\theta)$. If φ is a conditional, say $(\psi \rightarrow \theta)$, we'll set $\Im_{n+1}(\varphi)$ equal to the maximum of 1 - $\Im_n(\psi)$ and $\Im_n(\theta)$. If φ is a biconditional, say $(\psi\ \theta)$, we'll set $\Im_{n+1}(\varphi)$ equal to 1 if $\Im_n(\psi)$ and $\Im_n(\theta)$ are the same, and 0 if they are different. If φ is a negation, say ¬ψ, we'll set $\Im_{n+1}(\varphi) = 1 - \Im_n(\varphi)$.

Now set $\Im(\varphi)$ equal to $\Im_{degree\ of\ \varphi}(\varphi)$, for each sentence φ. It is easy to check that $\Im$ is a normal truth assignment.

Uniqueness. Suppose that H is a function assigning a value, either 0 or 1, to each atomic sentence and suppose that $\Im$ and K are both N.T.A.s extending H. We want to

prove that $\mathfrak{I} = K$ by using mathematical induction to show that, for each natural number n, for each sentence φ of degree n or less, $\mathfrak{I}(\varphi) = K(\varphi)$.

Base case: n=0. If φ has degree 0, φ is atomic, so $\mathfrak{I}(\varphi) = H(\varphi) = K(\varphi)$.

Inductive hypothesis: Now assume, that, for each sentence φ of degree n or less, $\mathfrak{I}(\varphi) = K(\varphi)$.

Inductive step: Let φ be a formula of degree n+1 or less. We want to see that $\mathfrak{I}(\varphi) = K(\varphi)$. If φ has degree less than n+1, this follows immediately from the inductive hypothesis. So we may assume φ has degree n+1. There are five cases:

Case 1: φ is a disjunction, say $(\psi \vee \theta)$. Then ψ and θ both have degree n or less. We have:

$\mathfrak{I}(\varphi) = 1$
iff $\mathfrak{I}(\psi) = 1$ or $\mathfrak{I}(\theta) = 1$ [because $\mathfrak{I}$ is a N.T.A.]
iff $K(\psi) = 1$ or $K(\theta) = 1$ [by inductive hypothesis]
iff $K(\varphi) = 1$ [because K is a N.T.A.]

Cases 2, 3, 4, and 5: φ is a conjunction, a conditional, a biconditional, or a negation. Similar.X

Second Proof: Let H be a function assigning a numerical value, either 0 or 1, to each atomic sentence. We'll define a *normal extension* of H to be a function I that assigns a numerical value, either 0 or 1, to each member of a subset of the set of SC sentences, and that meets the following six conditions:

(1)   If φ is an atomic sentence, φ is in the domain of I, and I$(\varphi) = H(\varphi)$.
(2)   If $(\varphi \vee \psi)$ is in the domain of I, then φ and ψ are both in the domain of I, and I$((\varphi \vee \psi))$ is the maximum of I(φ) and I(ψ).
(3)   If $(\varphi \wedge \psi)$ is in the domain of I, then φ and ψ are both in the domain of I, and I$((\varphi \wedge \psi))$ is the minimum of I(φ) and I(ψ).
(4)   If $(\varphi \rightarrow \psi)$ is in the domain of I, then φ and ψ are both in the domain of I, and I$((\varphi \rightarrow \psi))$ is the maximum of $(1 - I(\varphi))$ and I(ψ).
(5)   If $(\varphi \leftrightarrow \psi)$ is in the domain of I, then φ and ψ are both in the domain of I, and I$((\varphi \leftrightarrow \psi))$ is equal to 1 if I(φ) and I(ψ) are the same, and 0 if they're different.
(6)   If $\neg \varphi$ is in the domain of I, φ is in the domain of I, and I$(\neg \varphi) = 1 - I(\varphi)$.

Thus H is a normal extension of itself, and any NTA that extends H will be a normal extension of H. What we're aiming to show is that there is exactly one normal extension of H that includes all the SC sentences in its domain. The key fact about normal extensions is the following:

> **No-Conflict Lemma.** If J and K are normal extensions of H and φ is
> in the domain of both of them, then $J(\varphi) = K(\varphi)$.

Proof: We apply the Simplest-Example Lemma. If there were a sentence to which J and K assigned differing values, then there would be a simplest such sentence. Call it χ. There are six possibilities:

Case 1. χ is atomic. Can't happen. Since J and K both extend H, we have $J(\chi) = H(\chi) = K(\chi)$.

Case 2. χ is a disjunction, say $(\varphi \vee \psi)$. Since φ and ψ are both simpler than χ, we must have $J(\varphi) = K(\varphi)$ and $J(\psi) = K(\psi)$. Hence $J(\chi)$ = the greater of $J(\varphi)$ and $J(\psi)$ = the greater of $K(\varphi)$ and $K(\psi) = K(\psi)$. Contradiction.

Cases 3-6. χ is a conjunction, a conditional, a biconditional, or a negation. Similar to case 2.X

We now define a function I assigning numerical values, either 0 or 1, to the members of a set of SC sentences, as follows: A sentence is in the domain of χ iff it is in the domain of some normal extension of H. If K is a normal extension of H and φ is in the domain of K, we'll set Iφ) equal to $K(\varphi)$. We can safely stipulate this, because the No-Conflict Lemma assures us that different normal extension of H will never give us conflicting advice about what value to assign φ. In other words, I is the union of all the normal extensions of H.

It is easy to verify that I is a normal extension of H, by checking that it satisfies conditions (1) through (6). Thus I is the largest normal extension of H, a normal extension of H that includes every other.

> **Claim.** Every SC sentence is in the domain of I.

Proof: If not, then, according to the Simplest-Example Lemma, there is a simplest SC sentence outside the domain of I. Call it χ. There are six possibilities.

Case 1. χ is atomic. Impossible. Every atomic sentence is in the domain of H, which in a normal extension of H; so every atomic sentence is in the domain of I.

Case 2. χ is a disjunction, say $(\varphi \vee \psi)$. Because $(\varphi \vee \psi)$ is a simplest sentence not in the domain of I, φ and ψ must be in the domain of I. Form an extension I^+ of I by adding $(\varphi \vee \psi)$ to the domain, and setting $I^+((\varphi \vee \psi))$ equal to the greater of $I(\varphi)$ and $I(\psi)$. It is easy to verify, by checking conditions (1) through (6), that I^+ is a normal extension of I. But this is impossible, since I was constructed to be the largest normal extension of I.

Cases 3 - 6. χ is a conjunction, a conditional, a biconditional, or a negation. Similar to case 2.X

Thus I is a NTA extending H. This proves the existence part of the theorem, the claim that there is at least one NTA extending H. The uniqueness part – the claim that there is at most one NTA extending H – follows directly from the No-Conflict Lemma, since any NTA extending H will be a normal extension of H.<u>X</u>