HST.582J / 6.555J / 16.456J Biomedical Signal and Image Processing
Spring 2007

HST-582J/6.555J/16.456J-Biomedical Signal and Image Processing-Spring 2007

Problem Set 2

DUE: 2/22/07

Problem 1

(a) Let $h_{\text{lp}}[n]$ be the impulse response of an arbitrary lowpass filter. We form a new filter by multiplying the original filter's impulse response by a cosine, that is,

$$g[n] = h_{\text{lp}}[n] \cos(2\pi f_0 n).$$

Determine $G(f)$, the frequency response of $g[n]$, in terms of $H_{\text{lp}}(f)$, the frequency response of $h_{\text{lp}}[n]$.

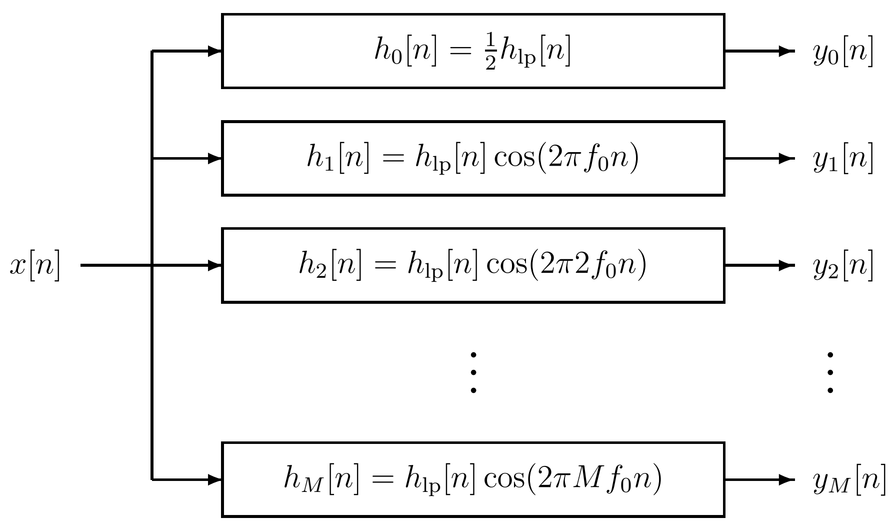

Figure 1:

Figure 1 shows a block diagram of a filter bank, where the filters have been designed using the technique demonstrated in part **(a)**. Filter banks are used in many applications to decompose a signal into components, or *bands*, corresponding to different frequency regions. The processing that follows the filter bank is typically a set of operations that gets performed on each band individually; in some applications the processed signals from all bands are then recombined to form a single system output.

It is often necessary to design the filter bank so that the input signal, $x[n]$, can be *exactly reconstructed* from the sum of the filter outputs,

$$y[n] = \sum_{m=0}^{M} y_m[n].$$

The transfer function between $x[n]$ and $y[n]$ is the *composite frequency response* of the filter bank.

Define $h_d[n]$, an ideal, zero-phase lowpass filter with cutoff frequency $\frac{1}{2}f_0$:

$$H_d(f) = \begin{cases} 1 & 0 < |f| < \frac{1}{2}f_0 \\ 0 & \frac{1}{2}f_0 < |f| < \frac{1}{2} \end{cases}$$

Furthermore, select the relationship between the bandwidth and the number of bands in the filter bank so that

$$f_0 = \frac{1}{2M+1}.$$

(b) Assume that $h_{\text{lp}}[n] = h_d[n]$. Sketch the frequency responses of the individual filters, $H_m(f)$ for $m = 0, \ldots, M$.

(c) Assume that we design $h_{\text{lp}}[n]$ using the windowing method ($h_{\text{lp}}[n] = w[n]h_d[n]$, where $w[n]$ is an arbitrary window centered at 0). Show that the input, $x[n]$, can be exactly reconstructed from the composite output, $y[n]$. Find an expression for $y[n]$ in terms of $x[n]$.

(d) Now consider the case where $h_{\text{lp}}[n]$ is an arbitrary lowpass filter. Does the resulting filter bank still provide exact reconstruction? Describe qualitatively how passband ripple, stopband ripple, finite transition bands, and non-linear phase affect the composite frequency response of the filter bank.

Problem 2

In this problem we will investigate the error associated with approximating a desired impulse response by windowing. We will show that the rectangular window minimizes the mean-square error of the frequency response.

Let $h_d[n]$ be the impulse response of the filter with desired frequency response $H_d(f)$. Let $h[n]$ be the impulse response of the FIR filter determined by windowing, that is,

$$h[n] = \begin{cases} h_d[n]w[n], & -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

The error in the frequency response, $E(f)$, is defined to be $E(f) = H_d(f) - H(f)$, where $H(f)$ is the DTFT of $h[n]$. The mean-square error, ϵ^2, is

$$\epsilon^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |E(f)|^2 \, df.$$

(a) Express the mean-square error, ϵ^2, in terms of $h_d[n]$ and $h[n]$. Hint: Use Parseval's theorem.

(b) Using the result of part **(a)**, show that the rectangular window minimizes ϵ^2. What is the minimum mean-square error?

(c) Despite the fact that the rectangular window minimizes the mean-squared error, other types of windows are often used. Discuss the factors that may make other windows preferable to the rectangular window in achieving particular filter specifications. In particular, consider the mainlobes and sidelobes of the windows in the frequency domain, and discuss how that will affect the resulting filters in terms of their transition bands, passbands, and stopbands.

Problem 3

In many applications, it is necessary to change the sampling rate of a discrete-time signal obtained from sampling a continuous-time signal. One possible approach would be to reconstruct the original continuous time signal and then resample at the new rate. However, for obvious reasons, we would prefer to do the processing with only discrete-time operations.

Increasing the sample rate of a discrete-time signal by an integer factor is accomplished by upsampling (or interpolation). Upsampling $x[n]$ by a factor of L consists of two stages. First a new sequence $x_1[n]$ is formed by inserting $L-1$ zeros between the original samples, that is,

$$x_1[n] = \begin{cases} x[\frac{n}{L}], & \frac{n}{L} \text{ is an integer} \\ 0, & \frac{n}{L} \text{ is not an integer.} \end{cases}$$

Then $x_1[n]$ is filtered with an ideal lowpass filter with cutoff frequency $f_c = \frac{1}{2L}$ to produce the upsampled signal, $x_u[n]$.

Reducing the sample rate by an integer factor is accomplished by downsampling (or decimation). Downsampling $y[n]$ by a factor of M also consists of two stages. First, $y[n]$ is filtered with an ideal lowpass filter with cutoff frequency $f_c = \frac{1}{2M}$ to produce $y_1[n]$. Then a new sequence is formed by extracting every M^{th} sample of the filtered signal, that is, $y_d[n] = y_1[nM]$.

(a) The signal $x[n]$ is upsampled by L as described above to produce $x_u[n]$. Determine $X_u(f)$ in terms of $X(f)$, the DTFT of $x[n]$. What is the function of the lowpass filter? (Consider its effect in the time domain).

(b) The signal $y[n]$ is downsampled by M as described above. Determine $Y_d(f)$ in terms of $Y(f)$. What is the function of the lowpass filter? (Consider its effect in the frequency domain).

(c) The sampling rate can be changed by a noninteger factor by performing upsampling followed by downsampling. The change in the sampling rate will be the ratio $\frac{L}{M}$. In this case, the two lowpass filters can be combined into one. What is the required cutoff frequency of the single lowpass filter in terms of L and M? Sketch the effect of upsampling followed by downsampling in the frequency domain for $L = 4$ and $M = 3$.

(d) Is the same result obtained if the two operations are interchanged, that is, if we downsample first and then upsample?