

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Departments of Electrical Engineering, Mechanical Engineering, and the Harvard-MIT Division
of Health Sciences and Technology

6.022J/2.792J/BEH.371J/HST542J: Quantitative Physiology: Organ Transport Systems

PROBLEM SET 3

SOLUTIONS

February 26, 2004

Problem 1

A microelectrode is inserted into a cardiac pacemaker cell, and the (schematized) potential recorded is shown in Figure 1. The intra-cellular and extra-cellular concentrations of sodium and potassium are shown in Table 1.

Figure 1:

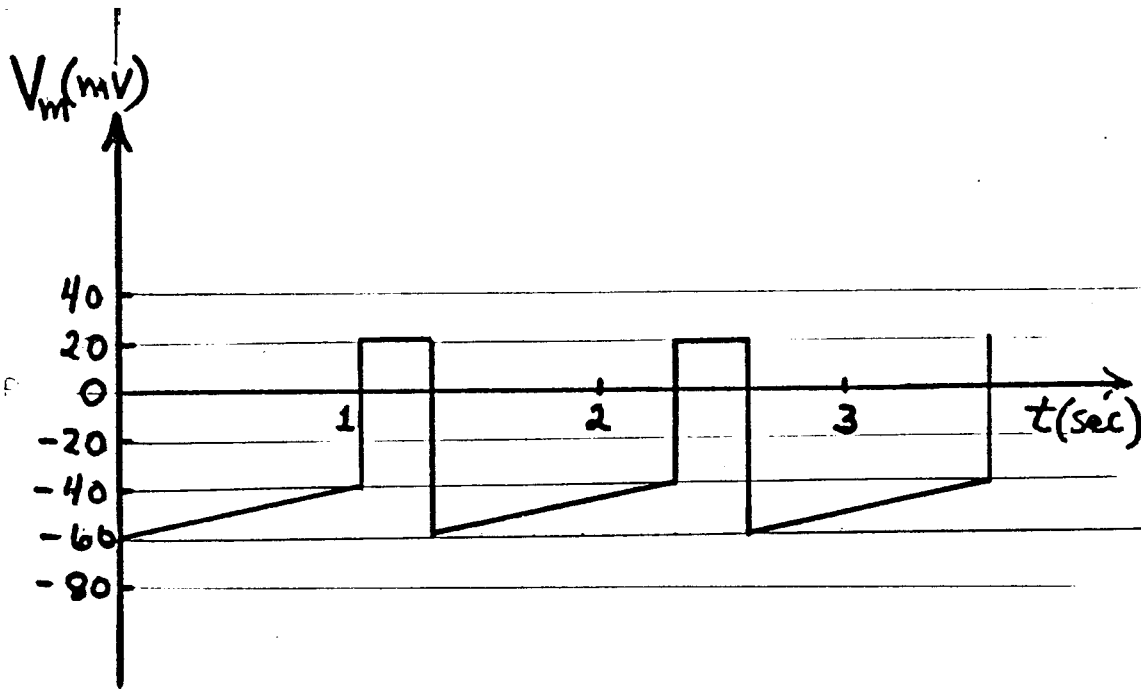


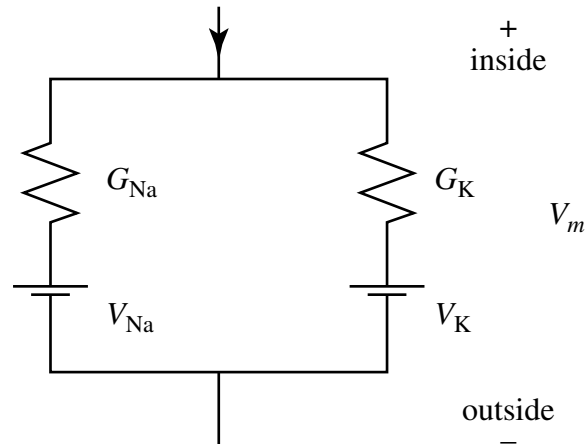
Table 1:

	$[K^+]$	$[Na^+]$
Inside	100	7
Outside	5	140

Questions:

- A. Assume that the membrane potential is determined *only* by the concentrations of Na⁺ and K⁺ and the membrane conductances to these ions, G_{Na} and G_{K} . Show a simple electrical model of the membrane, neglecting membrane capacitance. Label carefully, including polarity conventions and the inside and the outside of the membrane. Express V_m , the membrane potential, in terms of Nernst potentials (V_{Na} , V_{K}) and membrane conductances.

$$V_m = \frac{G_{\text{K}}}{G_{\text{K}} + G_{\text{Na}}} V_{\text{K}} + \frac{G_{\text{Na}}}{G_{\text{Na}} + G_{\text{K}}} V_{\text{Na}} \quad (\text{assuming } J_m = 0)$$



- B. What are the equilibrium potentials for sodium and potassium? Assume

$$\frac{RT}{F \log e} \approx 60 \text{ mV}$$

$$V_{\text{Na}} = 60 \log \left(\frac{[\text{Na}]_{\text{out}}}{[\text{Na}]_{\text{in}}} \right) = 60 \log \left(\frac{140}{7} \right) = 60 \log 20 = (60)(1.3) = +78 \text{ mV}$$

$$V_{\text{K}} = 60 \log \left(\frac{[\text{K}]_{\text{out}}}{[\text{K}]_{\text{in}}} \right) = 60 \log \left(\frac{5}{100} \right) = 60 \log .05 = (60)(-1.3) = -78 \text{ mV}$$

- C. Let α be the ratio of conductances of the membrane to potassium and sodium.

$$\alpha \equiv \frac{G_{\text{K}}}{G_{\text{Na}}}$$

Express the membrane potential V_m in terms of α .

$$\alpha = \frac{G_{\text{K}}}{G_{\text{Na}}}$$

$$\begin{aligned}
G_K + G_{Na} &= G_K + \frac{G_K}{\alpha} = G_K \left(1 + \frac{1}{\alpha}\right) = \frac{(1 + \alpha)}{\alpha} G_K \\
V_m &= \frac{G_K}{\left(\frac{1+\alpha}{\alpha}\right) G_K} V_K + \frac{\frac{G_K}{\alpha}}{\left(\frac{1+\alpha}{\alpha}\right) G_K} V_{Na} \\
&= \frac{\alpha}{(1 + \alpha)} V_K + \frac{1}{(1 + \alpha)} V_{Na} \\
&= \left(\frac{\alpha}{1 + \alpha}\right) (-78) + \left(\frac{1}{1 + \alpha}\right) (78) \\
&= \frac{78}{1 + \alpha} (1 - \alpha) = 78 \frac{(1 - \alpha)}{(1 + \alpha)}
\end{aligned}$$

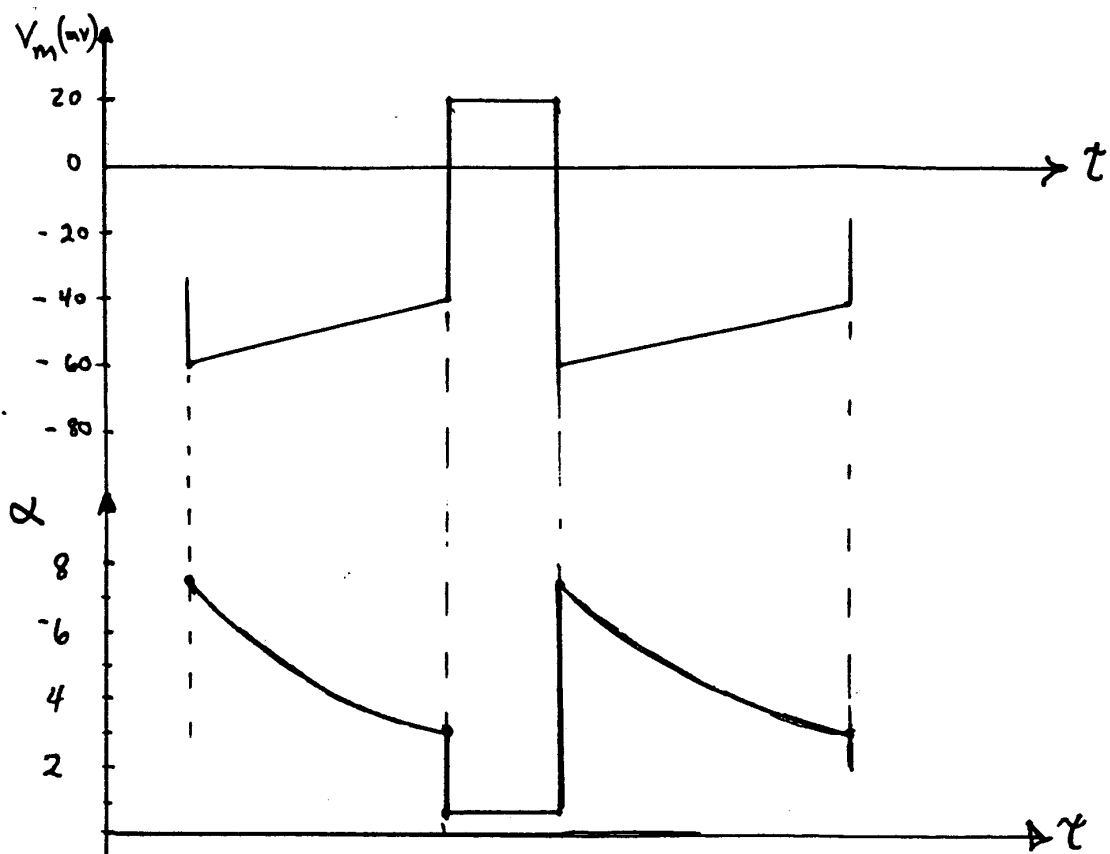
D. Sketch α vs. time for the cell.

Solve for α in terms of V_m .

$$\begin{aligned}
V_m &= 78 \frac{(1 - \alpha)}{(1 + \alpha)} \\
V_m + V_m \alpha &= 78 - 78 \alpha \\
(V_m + 78) \alpha &= (78 - V_m) \\
\alpha &= \frac{78 - V_m}{78 + V_m}
\end{aligned}$$

$$\begin{aligned}
V_m = -60 &\Rightarrow \alpha = \frac{78 + 60}{78 - 60} = \frac{138}{18} = 7.67 \\
V_m = -40 &\Rightarrow \alpha = \frac{78 + 40}{78 - 40} = \frac{118}{38} = 3.1 \\
V_m = +20 &\Rightarrow \alpha = \frac{78 - 20}{78 + 20} = \frac{58}{98} = 0.59
\end{aligned}$$

See sketch.

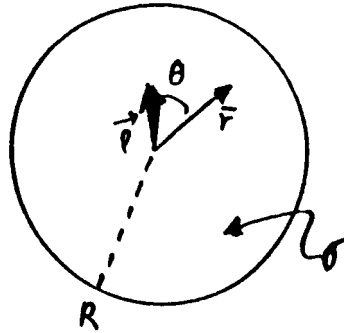


Problem 2

- A. Based on your understanding of the simple dipole model of electrocardiography, predict the effect of the following interventions on the QRS complex recorded from the standard lead I electrode connection.

Dipole Model:

Recall that the surface potential is

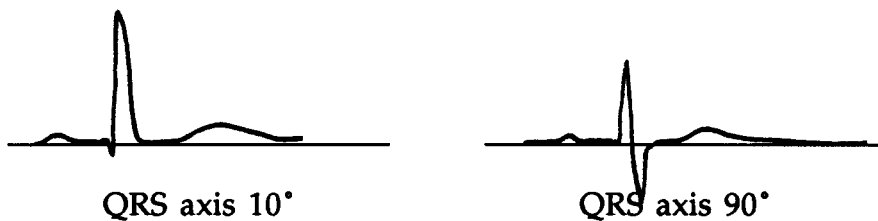


$$\Phi = \frac{3}{4\pi\sigma R^2} \cos \Theta$$

- (i) Increasing the radius of the torso by a factor of 2.
Measured potential will decrease by a factor of 4.
- (ii) Decreasing the conductivity of the tissue within the torso by a factor of 2.
Measured potential increases by a factor of 2.
- (iii) Changing the mean electrical axis of the heart from 10 to 90 degrees. (You may use a sketch if you wish).

When the mean QRS axis is 10 deg, the QRS vector is nearly parallel to Lead I. The QRS will be predominately positive in Lead I. When the QRS axis is 90 deg, the QRS vector is perpendicular to the Lead I. The QRS in lead I will have equally weighted positive and negative deflections, and usually will be smaller in amplitude.

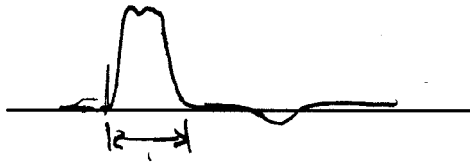
Lead I



- (iv) Triggering a ventricular depolarization via a pacemaker electrode in the RV, instead of via the normal conduction system.

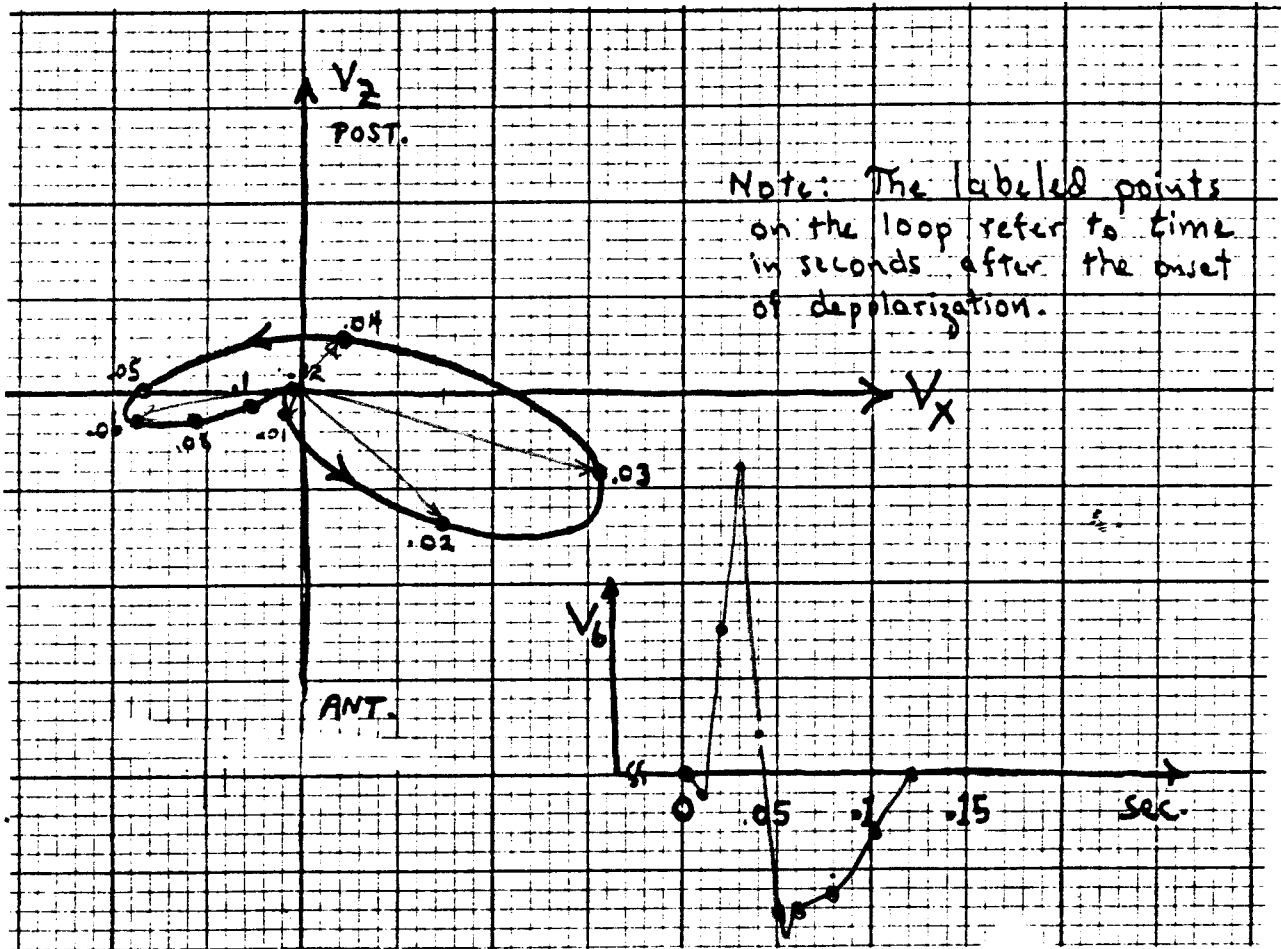
This will generate a wide ventricular depolarization complex (like a PVC) with a predominately positive deflection in lead I.

Lead I



Approx. 120 msec.

- B. From the horizontal plane VCG shown in the figure, sketch the expected scalar lead V-6 electrocardiogram. Use the axes provided. (The labeled points on the top indicate time in seconds following the onset of depolarization.)



Problem 3

Figures A through F show six scalar electrocardiograms. Figures 1 through 6 show *frontal plane* vectorcardiograms from the same six patients, but arranged in random order. Please unscramble them, and indicate the correct matches.

For each ECG also estimate the *mean electrical axis* in the frontal plane (in degrees).

ECG	VCG	Axis
A.	3	-130°
B.	5	80° <i>This is RBBB</i>
C.	1	-80° <i>(Long QRS duration, probably LBBB)</i>
D.	6	95°
E.	2	-30° <i>(LBBB)</i>
F.	4	0°

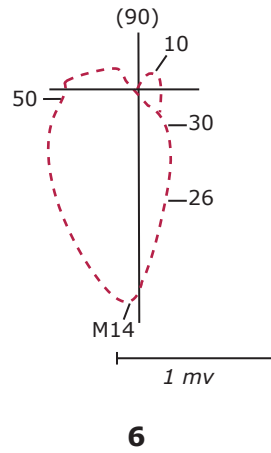
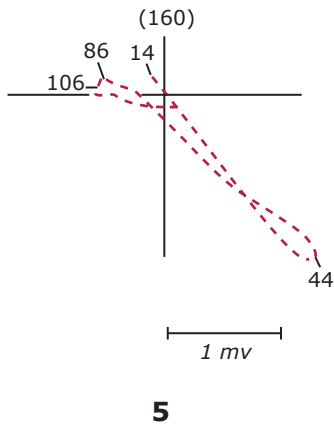
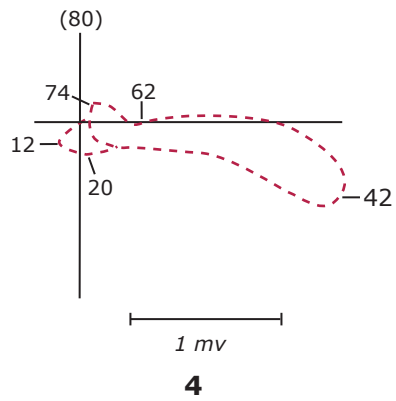
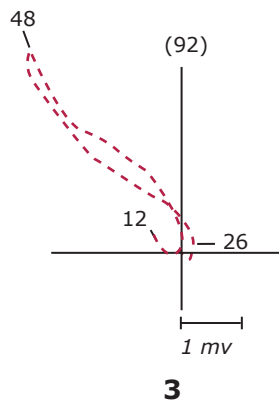
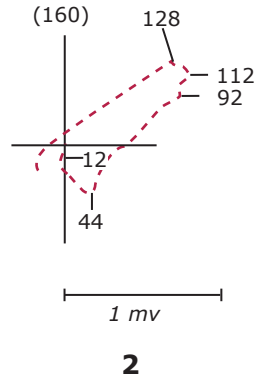
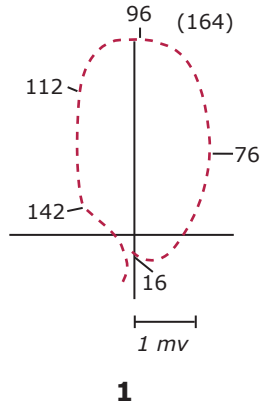
RBBB = right bundle branch block

LBBB = left bundle branch block

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Note: The number in parentheses indicates total QRS duration. The numbers on the VCG loops indicate the time (in msec.) after the onset of QRS.

2004/247

Problem 4

A new life-form has been discovered with an unusual cardiac anatomy shown in the attached figure. The creature has an ideal spherical torso in the center of which is located an interesting tubular heart with a helical shape. Depolarization begins at the top of the two-turn clockwise helix of radius R and pitch α . The action potential propagates at a constant velocity, v , along the tube until it reaches the outlet valve. The action potential is of long enough duration that repolarization does not begin until depolarization is complete. The action potential triggers a peristaltic contraction of the tube, which results in forward blood propulsion.

Figure 2: The creature, showing location of the helical heart.

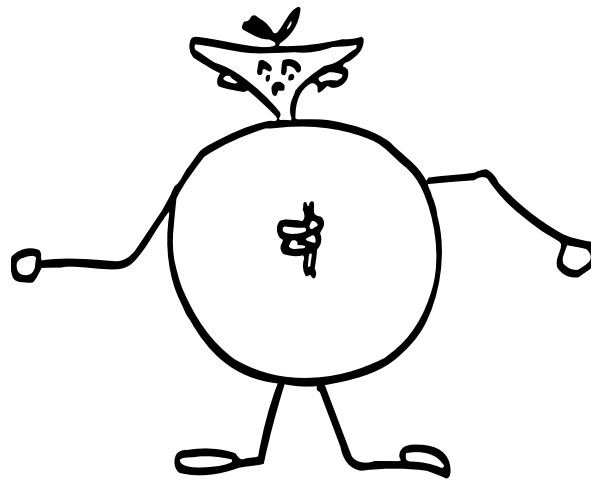


Figure 3: The tubular helical heart of radius R and pitch α .

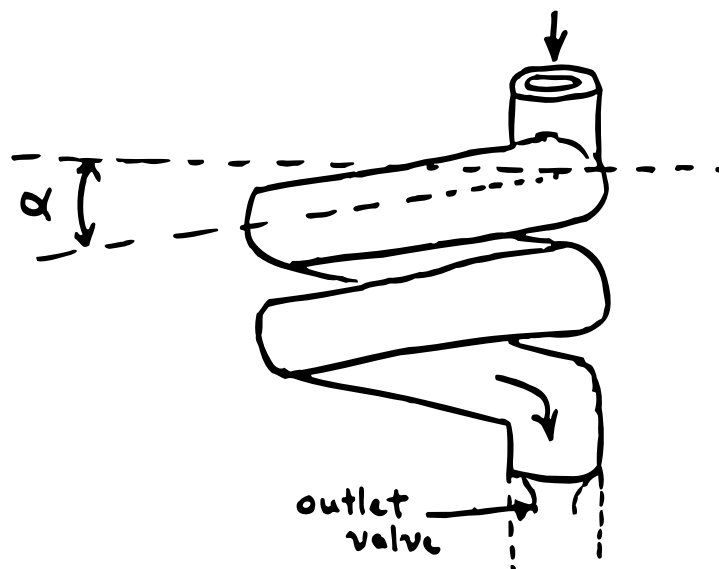
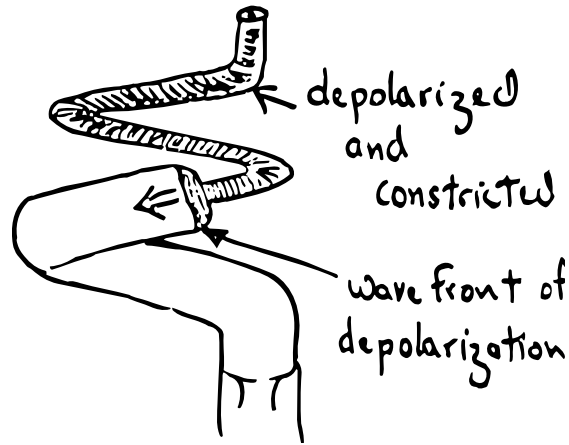
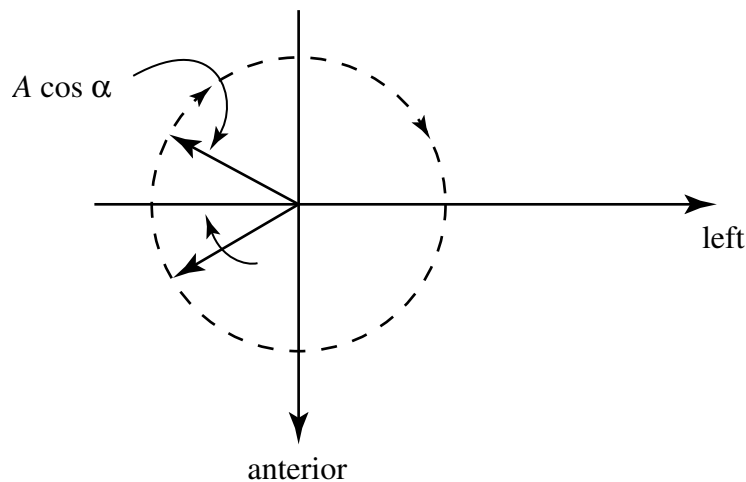


Figure 4: The heart during depolarization. Velocity of wave front propagation is v .

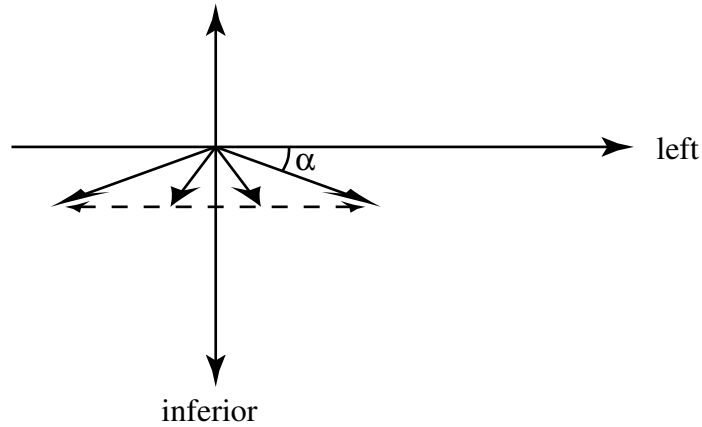


- A. Based on your understanding of the dipole theory of electrocardiography, sketch the expected ECG waveforms along the lead I axis, the aVF axis, and the third perpendicular axis projecting out of the chest (V2). Consider only the depolarization waveform.

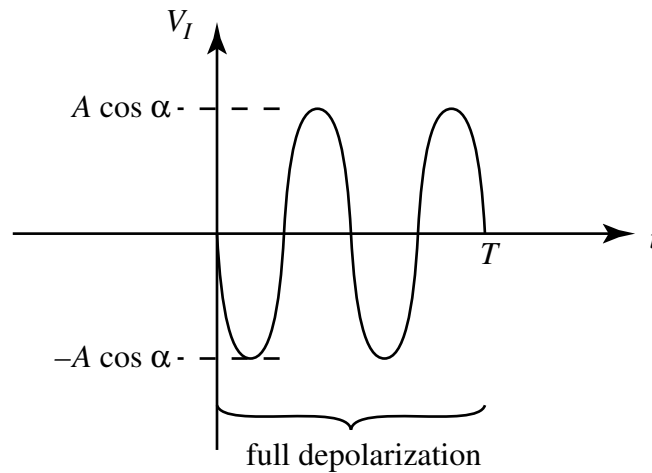
The heart dipole will begin by pointing directly anteriorly, and then the locus of the tip of the dipole will inscribe two complete circles of constant amplitude when viewed from above (the horizontal VCG projection):



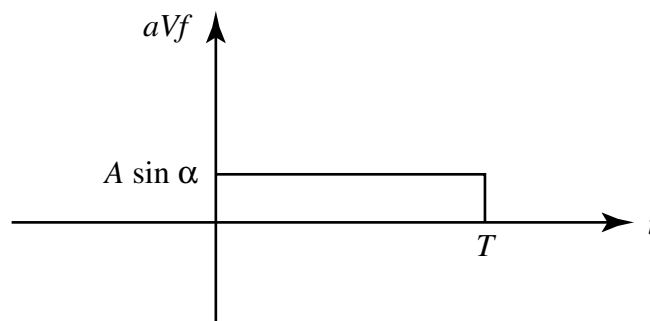
If the magnitude of the heart vector is A , then the magnitude of the vector projected into the horizontal plane is $A \cos \alpha$. The heart vector will point downward at an angle α so the frontal plane VCG would be as follows:



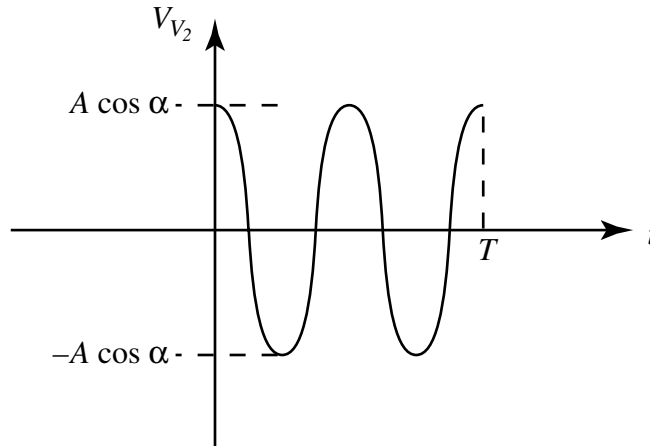
Thus, the projection on Lead I (which is directed from right shoulder to left shoulder) would be sinusoidal:



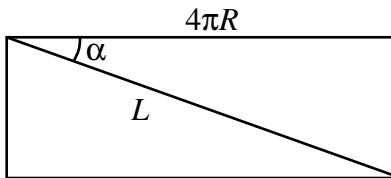
The projection of the heart vector on lead avF would be constant for the duration of depolarization, and of value $A \sin \alpha$.



The projection on an axis projecting out of the chest would also be sinusoidal, as for the lead I projection, and initially at full positive amplitude.



The duration of the complex, T , would be approximately $4\pi R/v$.
 A more precise calculation of duration includes the pitch.



$$L = \text{total heart length}$$

$$\cos \alpha = \frac{4\pi R}{L}$$

$$L = \frac{4\pi R}{\cos \alpha}$$

$$T = \frac{L}{V} = \frac{4\pi R}{v \cos \alpha}$$

B. What would be the change in the amplitude of the ECG if:

The body surface potentials are given by

$$\phi(\theta)_{\text{surface}} = \left[\frac{3|M|}{4\pi\sigma R^2} \right] \cos\theta$$

(i) The radius of the torso were doubled?

If the torso radius, R , is doubled, the ECG amplitude would decrease by a factor of 4.

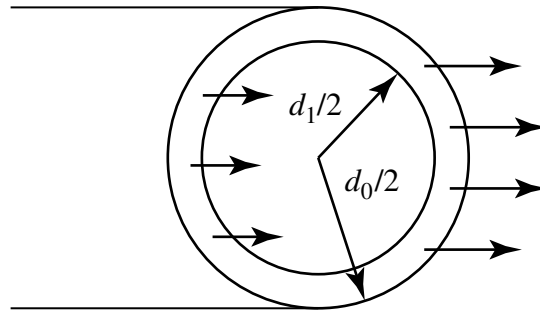
(ii) The electrical conductivity of the torso were doubled?

If the conductivity, σ , is doubled, ECG amplitude would decrease by a factor of 2.

C. Estimate the peak-to-peak amplitude (in mV) of the “QRS” complex recorded in lead I. Base your estimate on the following considerations, and use the same assumptions presented in the class notes.

- The outer diameter of the tubular heart is 0.5 cm., and the inner diameter is 0.3 cm. (The radius of the helix, R , is 2 cm.)
- The turns of the relaxed heart tube are touching one another.
- Individual myocardial cells are oriented longitudinally along the tube and have a diameter of 10 microns.
- The internal resistance per unit length of the cells, r_l , is 10^8 ohms/cm. The external resistance, r_0 , is much smaller, and may be neglected.
- The action potential morphology is similar to that of human ventricular cells, and has a phase 0 amplitude of 100 mV and a rise-time of 1 msec.
- The velocity of propagation is 100 cm/sec.
- The torso radius is 10 cm and its conductivity is 1×10^{-3} mho/cm.

The peak-to-peak amplitude would be twice the maximum projection of the heart vector on lead I.



First, how many individual heart dipoles contribute to the net heart current dipole?

$$N = \frac{(\text{cross-sectional area of the tube})}{(\text{cross-sectional area of individual cell})} = \frac{A_T}{A_c}$$

$$A_T = \frac{\pi}{4}(d_0^2 - d_1^2) \approx \frac{3}{4}(0.5^2 - 0.3^2) = \frac{3}{4}(0.16) = 0.12 \text{ cm}^2$$

$$A_c = \frac{\pi d^2}{4} \approx \frac{3}{4}(10 \times 10^{-4})^2 = \frac{3}{4}(10^{-6}) = 0.75 \times 10^{-6} \text{ cm}^2$$

$$N = \frac{(0.12)}{(0.75 \times 10^{-6})} = 0.16 \times 10^6 = 1.6 \times 10^5$$

current in one cell:

$$i = \frac{1}{(r_i + r_0)} \cdot \frac{1}{c} \cdot \partial V_m(x, t) \partial t \quad \text{where } c \text{ is the velocity of propagation}$$

$$r_i \gg r_0, \quad r_i = 10^8 \Omega/\text{cm}$$

$$c = 100 \text{ cm/sec}$$

$$\begin{aligned} \text{derivative of } V: \quad \frac{\partial V}{\partial t} &= \frac{100 \text{ mV}}{10^{-3} \text{ sec}} \\ &= \frac{100 \times 10^{-3} \text{ V}}{10^{-3} \text{ sec}} = 100 \text{ V/sec} \end{aligned}$$

$$\begin{aligned} \text{current through one cell: } i &= 10^{-8} \frac{\text{cm}}{\Omega} \cdot \frac{\text{sec}}{100 \text{ cm}} \cdot \frac{100 \text{ V}}{\text{sec}} \\ &= 10^{-8} \frac{\text{V}}{\Omega} = 10^{-8} \text{ A} \end{aligned}$$

$$\text{length of depolarized region} = l$$

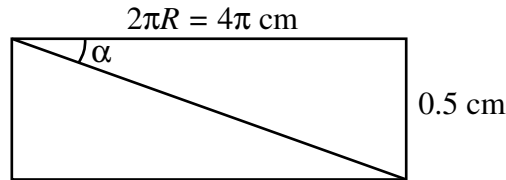
$$l = 100 \text{ cm/sec} \times 1 \times 10^{-3} \text{ sec} = 0.1 \text{ cm}$$

$$\rightarrow m = l \times i = 10^{-8} \text{ A} \times 0.1 \text{ cm} = 10^{-9} \text{ A cm}$$

m is the dipole for each cell; the total dipole

$$M = Nm = 1.6 \times 10^5 \times 10^{-9} \text{ A cm} = 1.6 \times 10^{-4} \text{ A cm}$$

$$V = \Phi_{LA} - \Phi_{RA} = (\vec{M} \bullet \vec{L}) = |M| \cdot 2 \cos \alpha \cdot |OA| \quad (\text{twice the amplitude for peak to peak})$$



$$\tan \alpha = \frac{0.5}{4\pi} = \frac{1}{8\pi}$$

$$\alpha = \tan^{-1} \left(\frac{1}{8\pi} \right)$$

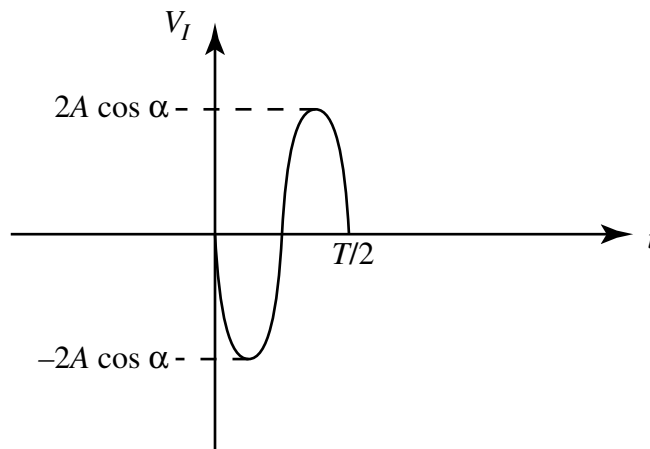
$$|OA| = \frac{3}{4\pi \sigma R^2}$$

Thus,

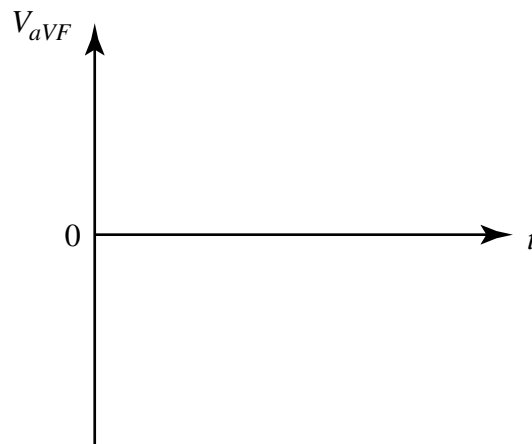
$$V = (1.6 \times 10^{-4} \text{ A cm}) \left(2 \cos \left[\tan^{-1} \left(\frac{1}{8\pi} \right) \right] \right) \left(\frac{3}{4\pi} \right) \left(\frac{1}{\frac{10^{-3}}{\Omega} (10 \text{ cm})^2} \right) \approx 0.7 \text{ mV}$$

- D. An ectopic beat originates in the exact middle of the heart, at exactly one turn in from the top and from the bottom. Sketch the expected ECG waveforms resulting from the ectopic beat along the same three axes used in part a (I, aVF, V2).

The ectopic beat will result in two depolarizing wavefronts travelling in opposite directions around the helix. The 2 waveforms move in the same direction at the same time when projected on the lead I axis, resulting in twice the normal amplitude for one cycle.



The two wavefronts move in opposite directions at the same time when projected on the aVF lead, so there is no net heart dipole projected on the aVF lead.



Similarly, there is no amplitude on the V_2 lead.

