III. Measuring the Unmeasurable: Putting the Irrational in Ratio

1]] In *Topics* 158b29 Aristotle states that when a line parallel to the base cuts a triangle, it divides the sides and the areas similarly, "…for the areas and the straight lines have the same *antanairesis*, and this is the definition of 'the same ratio'." The process of *antanairesis* (called by Euclid *anthuphairesis*) is the process used to find the Greatest Common Divisor in Book X.1. How does this process 'define' a ratio? How does this definition compare with the definitions of ratio in Book V.

2] How do you think the Greeks came to suspect that two lines could have no common measure (hint: think 'Pythagorean')?

3] Thinking about the 'Pythagorean' proof of the incommensurability of the diagonal and the side of the square (or, as we might put it, of the $\sqrt{2}$), Can you use the same method to prove that $\sqrt{3}$ and $\sqrt{5}$ are irrational? Why doesn't this kind of proof demonstrate that $\sqrt{4}$ is irrational (answer is not 'because it isn't)?

4] Given that the Greeks often substituted for the word 'arithmos' the phrase 'the odd and the even', as if these mean the same thing, what might be the significance of proving that any ratio of integers expressing the relationship of the side to the diagonal would have to be both even?

5] In Euclid, Book. V: What is it about magnitudes that are commensurable in square that makes them 'expressible' in terms of each other? Why does this not hold, for instance, for magnitudes commensurable in cube?

6] Proposition X.2 is never used by Euclid to discover or prove incommensurability. In fact, it is used incidentally in Proposition X.3, but then never again employed by Euclid. What then is the function of this Proposition? Is it a kind of definition? If so, is it an <u>essential</u> (*ti esti*) definition, or a definition by way of identifying a necessary attribute of incommensurables (that is, a definition by exclusion)?

7] How could Proposition XIII.5 be converted into a proof of the incommensurability of the segments of the 'golden ratio.'(Hint: start performing an *anthuphairesis* between the final, larger line and its larger segment)

8] When we reverse the anthuphairetic proof of the incommensurability of the side and diagonal of the square, the ratios of the sides and diagonals seem to approach the $\sqrt{2}$. Can you prove that this progression of ratios is approaching the true value of the side/diagonal ratio (and can you do so in a "Greek", i.e. geometric manner)?

9] 'Why' does reversing the anthuphairesis give us this approximating progression?

10] The 'common measure' of the side and diagonal, if it exists in any fashion, seems to be an entity of which we can always say, with great precision, whether it is greater or less than any particular arithmos, but can never say with precision what exactly it is. Thinking back to

Aristotle's categories in Chapter 2, have we grasped this entity as a 'quantity' or as a 'relation', or what?

11] The pattern generated by those incommensurable magnitudes that can be proven incommensurable through anthuphairesis is palindromic. Can you see why this is so, given that the reverse of the anthuphairesis proving them to be incommensurable, always yields a series of ratios successively approximating the ratio of the two incommensurable?

12] In Book X, are the medial, apotome, and the binonomial in any way 'essentially' different?

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