## B. Logos as Definition:  What is "What is?"

1] When you give a definition of 'straight line', what are you trying to define? [i.e. how can you state what you are trying to define without in fact defining it?]

2] What does Euclid's definition of a straight line mean?

3] I have in my possession a high school geometry text that I found at the town dump.  On the inside cover it has the name Charles P. Drew written, and then, right under the warning not to write in the book is written, in a woman's handwriting, 'Hi Charlie!' with the dots over the i's in the shape of hearts.  The textbook says the following about definitions:  "A **definition** uses known words to describe a new word.  In geometry, some words, such as point, line, and plane, are undefined terms.  Although these words are not formally defined, it is important to have general agreement about what each word means.  A **point** has no dimension.  It is usually represented by a small dot.  A **line** extends in one dimension.  It is usually represented by a straight line with two arrowheads…A **plane** extends in two dimensions.  It is usually represented by a shape that looks like a tabletop…"  Compare these definitions to Euclid's, and this 'theory of definition' to Aristotle's.  Is there a 'modern/ancient' distinction here?

4] Go back through *Meno*, list all the definitions of anything, and create categories of types of definitions.  Use Aristotle to categorize and/or make up your own.

5] Thinking of Plato and Bacon, if the ancient and modern account of knowledge is quite different, what is it that they are disagreeing about?  That is what is this 'knowledge' the nature of which is in dispute?

6.] <u>For those familiar with Non-Euclidean geometry</u>:  For those who deny the postulates that create Euclidean space, there is an argument about the nature of a straight line.  If we accept the postulates of spherical geometry (that is, not merely conceiving of geometry on a sphere, within Euclidean space, but imagining that geometry on what we perceive to be a plane has the characteristics of geometry on a sphere) then we will assign different properties to straight lines and their relationships (e.g. the possibility of parallel lines).  If we are arguing about what the properties of a straight line are, what is the object about which we both are disagreeing?  If we can come up with a definition of this object (it's not too hard) that applies in both realms, is that, then, necessarily the most 'essential' definition?

7] Why does Postulate 4 need to be stated?  How is it other than the principle of identity ("All angles of the same measure are of the same measure")?

8] Aristotle distinguishes between definitions based on what is better known to us, and definitions which are based on what is better known (i.e. more *knowable*) absolutely.  Of which sort are Euclid's definitions?

9] Aristotle claims [*Posterior Analytics* 90a31]:  "to know what a thing is [ti estin] is the same as knowing *why* it is [dia to estin]".  How could this apply to Euclid's definitions of mathematical objects?

# C. The Idea of a Proof

1] Smith reports "The idea of demonstrating the truth of a proposition which had been discovered intuitively appears first in the teachings of Thales (c. 600 B.C.)  It is probable that this pioneer knew and proved about six theorems, each of which would have been perfectly obvious to anyone without any demonstration whatever."  Proofs are not found anywhere in antiquity, except in Greek intellectual life.  Speculate: what would cause someone to prove something that is perfectly obvious, and thereby inaugurate a system in which the non-obvious could be proven, especially given that the entire rest of the world, for the 1000 years following Thales, were perfectly content to operate with empirically reliable, unproven theorems?

2] Euclid's Proposition 6, proposition 14, and proposition 19 [Bk. I] and others are negative or 'reductio ad absurdam' proofs.  There is currently a school of mathematical thought [constructivism] that rejects the validity of such proofs [can you guess why?]  At very least, it is generally considered more enlightening to prove something directly.  What forces Euclid to resort to a 'reductio' proof? [Think about what the reduction proofs in Euclid have in common]. [Is I.48 a reductio proof in disguise?]

3] What are the conditions of a reasonable axiom? What constitutes a reasonable 'common notion'?

4] **Analysis and Synthesis:**
**A]** Anonymous scholium to Euclid XIII:

"Analysis, then, is the taking of what is sought as if it were agreed to be true and following through the consequences from this to something genuinely agreed to be true, while synthesis is the taking of something admitted and going through to something we then agree to be true." [I took some liberties with translation to make this clearer to you. Apologies to Anonymous]

**B]** Pappus of Alexandria [neo-Platonist, 4th century A.D.]:

"... in analysis we suppose that which is sought to be already done, and inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle... But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought..."

"Analysis, then, is the way from what is sought, taken as admitted by means of previous synthesis…but in synthesis, going in reverse, we suppose as admitted what was the last result of the analysis, and, arranging in their natural order as consequences what were formerly the antecedents, and connecting them with one another, we arrive at the completion of the construction of what was sought, and call it synthesis."

**C]** Francois Viete (the true founder of Western algebra, 16th century): "In mathematics there is a certain way of seeking the truth, a way which Plato is said first to have discovered, and which was called 'analysis' by Theon and was defined by him as 'taking the thing sought as granted and proceeding by means of what follows to a truth that is uncontested'; so, on the other hand, 'synthesis', is 'taking the thing that is granted and proceeding by means of what follow to the conclusion and comprehension of the thing sought

**D]** Rene Descartes (16th century) in *Objections and Replies*:
"…the method of proof is two-fold, one being analytic, the other synthetic. Analysis shows the true way by which a thing was methodically discovered and derived, as it were effect from cause, so that, if the reader care to follow it and give sufficient attention to everything, he understands the matter no less perfectly and makes it as much his own as if he had himself discovered it. But it contains nothing to incite belief in an inattentive or hostile reader; for if the very least thing brought forward escapes his notice, the necessity of the conclusions is lost; and on many matters which, nevertheless, should be specially noted it often scarcely touches, because they are clear to anyone who gives sufficient attention to them.

"Synthesis contrariwise employs an opposite procedure, one in which the search goes as it were from effect to cause (though often here the proof itself is from cause to effect to a greater extent than in the former case). It does indeed clearly demonstrate its conclusions, and it employs a long series of definitions, postulates, axioms, theorems and problems, so that if one of the conclusions that follow is denied, it may at once be shown to be contained in what has gone before. Thus the reader, however hostile and obstinate, is compelled to render his assent. Yet this method is not as satisfactory as the other and does not equally well content the eager learner, because it does not show the way in which the matter taught was discovered.

"It was synthesis alone that the ancient Geometers employed in their writings, not because they were wholly ignorant of the analytic method, but, in my opinion, because they set so high a value on it that they wished to keep it to themselves as an important secret."

     **1]** Which, of any, of Euclid's proofs are analytic?

     **2]** Is the mathematical passage in *Meno* analytic or synthetic?

     **3]** Rewrite Euclid II.4 as an analytic proof. [You are excused, for this exercise, from the general prohibition against using modern mathematics or notation]

5] Any valid proof demonstrates *that*, given certain premises, something is the case. Do some proofs also show *how* or even *why* something is the case? That is, do some proofs not only demonstrate, but explain? (Proclus, for instance, criticizing I.23 offers an alternative construction he claims proceeds in a 'more instructive manner') What are the characteristics of an explanatory proof, or what makes a proof more or less explanatory? Which proof in Book I do you think provides the most 'insight' into its subject matter?

6] Why is I.10, written as a construction problem rather than the proposition "The line bisecting the angle of an equilateral triangle bisects the opposite side?" In general, what makes him

choose to present a conclusion as a construction or as a theorem [Q.E.D. or Q.E.F.]?

7] Work out a classification scheme for the proofs we read for this assignment.

8] Think of the *Elements* as a work of literature. Heath says that Book I has three parts. Until Proposition 27, it is a book about triangles. Propositions 27-31 are about parallel lines. The rest of the book is about parallelograms and their relationship to triangles. What is the 'story line' of this book? **C.I. presentation: choose one proposition to present in class, and tell how it fits into the story that Book I tells.**

9] Proposition 27, which shows that two lines intersected by a transverse line, making equal alternate angles with the transverse, will never meet. This proposition does not draw in any way on Postulate 5, the famously misnomered 'parallel postulate'. We all know that alternate geometries can be formulated in which either [a] all lines meet, or in which [b] a number of lines drawn through any point are parallel to a third line ([b] is not ruled out in Euclid until Proposition 29, which *does* cite postulate 5). So what is it that has created 'Euclidean space' in the *Elements*, prior to I.27, if not the 'parallel postulate'?

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